A THEORETICAL LOOK AT PIXEL ORDERING

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Summary

The performance of predictive coding techniques used in most lossless image compression algorithms depends very much on the pixels (neighbors) used for prediction. This thesis studies the pixel ordering problem – how pixel ordering and pixel choice affect the compression in predictive coding.

We discuss briefly the statistical assumptions of predictive coding and its application to derive a mathematical measure to quantify how bad a given pixel ordering and pixel choice combination can be. This measure is applied to a class of raster scan pixel ordering and a few pixel orderings based on the Multi-Level Progressive (MLP) method by Howard and Vitter. Although simpler than the MLP, the raster scan pixel ordering proved to be better under our measure. This result is further corroborated by our experiments.

The measure mentioned above implicitly assumes a prediction function that treats each neighbor equally; however, not all prediction functions do so in reality. The effect of the weights in a prediction function on pixel ordering is therefore investigated and an improved badness measure conditioned upon optimal weights derived. The above-mentioned pixel orderings are analyzed again with this measure without much change to their relative performance.

Chapter 1

Introduction

Images have become an important part of electronic communication. From surveillance systems to the graphics on Internet web pages, images are stored electronically and transmitted over wired networks, satellite links and other media. Although storage is cheap, bandwidth is usually not; hence the importance of efficient electronic representation of images.

Storing a 512 by 512, 8 bits per pixel greyscale image without compression requires $512 \times 512 \times 8$ bits. With compression we can do better. Data compression is basically assigning short codes for long ones such that the original data can be recovered from the compressed data. Image compression is applying data compression upon a particular type of data – images. The difference between data compression and image compression is that image compression usually makes use of certain properties particular to images to achieve more compression.

Some image compression algorithms do not recover the exact original im-

ages upon decompression; such algorithms are called lossy compression algorithms and those that recover the exact original are described as lossless. One common technique in lossless image compression algorithms is to process each pixel of an image in some order and predict each pixel using previously encoded/decoded pixels. The *prediction error*, the difference between the predicted value and the actual pixel value, is then encoded. This is also known as *predictive coding*. For more detailed descriptions, the reader may like to consult [Sal98].

This thesis is mainly concerned with how *pixel ordering* and *pixel choice* that are used in the prediction step affect compression. **Pixel ordering** is defined to be the order or sequence in which an image is scanned and processed at the prediction step and **pixel choice** is the selection of previously processed pixels (*neighbors*) used for prediction. Pixel ordering and pixel choice are mutually constraining: a particular pixel ordering will limit which pixels can be used in the pixel choice and vice versa. Hence is it difficult to consider their effect on compression in separately. We loosely call this the **pixel ordering problem**. Questions that we try to address are: How can the pixel ordering of different algorithms be evaluated? Is there a particular sequence/ordering of neighbors that gives particularly good compression? We attempt to answer these questions by proposing a mathematical measure of how bad a given pixel ordering can be and analyze the pixel ordering of two compression methods: the *Multi-Level Progressive* (MLP) method and the raster scan method. Greyscale images are assumed for clarity, but the results can be easily extended to color images.

The next chapter will introduce this badness measure and discuss related

issues. The following chapters will then present the detailed analysis of MLP and raster scan pixel orderings. Chapter 5 will propose a way of dealing with weighted neighbors. Following which we will analyze the two pixel orderings again using optimal weights. The last chapter will conclude this thesis with a discussion of what has been achieved in this thesis and propose some future directions.

Chapter 2

The Badness Measure

In this chapter, we first discuss the statistical background that motivated our new badness measure, derive and present the badness measure and discuss some of the issues relating to the parameters of our badness measure. We end off with an explanation of the notation and conventions used in the rest of this thesis.

2.1 Motivation

A common statistical measure of compression is the entropy, given by,

$$H_0 = -\sum^{\text{all symbols i}} P(X = i) \log P(X = i) \tag{2.1.1}$$

which measures the average number of bits used to reversibly encode a symbol in the uncompressed data. For image compression, the entropy can be interpreted as the average bits per pixel of the compressed image. This is be-

cause the symbols are now pixel values or the compressed codes representing the pixel values. The probabilities in the calculation of entropy can be obtained by counting the frequency of the occurrences of each encoded value or by assuming a probability distribution and calculating the probability of the encoded value as if it were drawn from that distribution.

For predictive coding methods, the prediction errors and not the pixels are encoded. The aim of predictive coding is to transform a greyscale image into an error image such that the error image has less number of different values to code. For example, a prediction function that predicts a n pixels 8-bit greyscale image perfectly will generate an error image of all zeroes and hence of zero variance. To encode this error image (consisting of n zeroes) requires only an average of $\frac{1}{n}\log n$ bits per pixel. Consider another example where the prediction function transforms the image into an error image consisting of 2 values, say 0 and 1 only. Effectively, this predictor has transformed the 8-bit image with 256 grey levels into a 1-bit error image. Let p_0 and p_1 be the proportion of the errors that are of the values 0 and 1 respectively. The average number of bits to encode a prediction error would then be $-p_0 \log_2 p_0 - p_1 \log_2 p_1$. Another way to interpret this expression is $-\sum_{all\ n\ errors}\left(\frac{1}{n}\right)\log_2 P(error\ value)$ which looks like the average of all the '$-\log_2 P(error\ value)$'.

From this simple example, it should be clear that the encoding of the error image depends on the probability distribution of the error values. It has been found that the distribution of the error values resembles a zero mean Laplacian distribution [O'N66] whose probability density function is given by:

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\sqrt{\frac{2}{\sigma^2}}|x-\mu|\right) \tag{2.1.2}$$

where μ is the mean (which is zero in this case) and σ^2 is the variance. The integral of this expression will give the discrete probability mass function,

$$P_{\sigma^2}(k) = \int_{k-0.5}^{k+0.5} \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\sqrt{\frac{2}{\sigma^2}}|x-0|\right) dx. \tag{2.1.3}$$

It should be easy to see that $log_2 P(error\ value) \propto |error\ value|$ and hence the total number of bits required to code the error image is proportional to the sum of all the absolute error values.

The badness measure that we propose in the next section is meant to be the sum of the estimates of the expected absolute error value over the entire image. The relationship between the average absolute error and the entropy of the system will allow our proposed measure to assess how compressible the prediction errors resulting from a pixel ordering are.

2.2 The New Badness Measure

Suppose $X_1, X_2, \ldots X_c$ are the random variables for the pixel values of the c neighbors used in an arbitrary prediction. We will assume that the X_i's are independent. Let Y be the true pixel value of the predicted pixel and $\hat{Y}$ the predicted value.

Since the prediction errors are found to be of zero mean experimentally,

$$E(\hat{Y} - Y) = 0 \tag{2.2.4}$$

$$E(\hat{Y}) - Y = 0 \tag{2.2.5}$$

$$E(\hat{Y}) = Y. \tag{2.2.6}$$

It is difficult to obtain an expression for an arbitrary prediction error without assuming some prediction function. For simplicity, the mean is assumed. This assumption will be re-examined in Chapter 5. Hence,

$$\hat{Y} - Y = E(X_i) - Y \tag{2.2.7}$$

$$= \frac{1}{c} \sum_{i=1}^{c} (X_i) - Y. \tag{2.2.8}$$

Since Y is constant with respect to i,

$$\hat{Y} - Y = \frac{1}{c} \sum_{i=1}^{c} (X_i - Y). \tag{2.2.9}$$

Our badness measure is defined to be the sum of the expected absolute error of every prediction; however, the expected absolute error is difficult to work with and we estimate it with the square root of the mean squared error,

$$E\left(\left|\hat{Y} - Y\right|\right) \approx \sqrt{E\left[\left(\hat{Y} - Y\right)^2\right]}. \tag{2.2.10}$$

But $Y = E(\hat{Y})$ implies $E[(\hat{Y} - Y)^2] = Var(\hat{Y})$,

$$E\left(\left|\hat{Y} - Y\right|\right) \approx \sqrt{Var\left(\hat{Y}\right)} \tag{2.2.11}$$

$$= \sqrt{Var\left(\hat{Y} - Y\right)}. \tag{2.2.12}$$

Substituting Equation 2.2.9,

$$E\left(\left|\hat{Y} - Y\right|\right) \approx \sqrt{Var\left[\frac{1}{c} \sum_{i=1}^{c} (X_i - Y)\right]}. \tag{2.2.13}$$

Assuming $(X_i - Y)$'s are independent,

$$E\left(\left|\hat{Y} - Y\right|\right) \approx \sqrt{\frac{1}{c^2}\sum_{i=1}^{c} Var\left(X_i - Y\right)} \tag{2.2.14}$$

$$= \frac{1}{c}\sqrt{\sum_{i=1}^{c} Var\left(X_i - Y\right)}. \tag{2.2.15}$$

Using the above result, we propose our 'badness' measure for evaluating pixel ordering as the sum over all pixels of absolute error estimates,

$$\text{Badness } \mathcal{B} = \sum_{\text{all pixels}} \begin{cases} z' & \text{no prediction} \\ \frac{\sqrt{\sum_d c_d f(d)}}{\sum_d c_d} & \text{otherwise} \end{cases} \tag{2.2.16}$$

where d's are the Manhattan distances of all the neighbors used in the prediction of the current pixel, c_d is the number of distance d neighbors used in the prediction and $f(d)$ is a function that increases with increasing distance. $f(d)$ is basically an estimate for $Var\left(X_i - Y\right)$. z' is the cost of encoding a pixel instead of its prediction error and is used to account for pixels that cannot be predicted such as the starting pixels transmitted to be used as neighbors for the very first prediction.

This badness measure has the following two desirable properties: neighbors further away are worse predictors than neighbors that are nearer; and, using more neighbors reduces prediction errors.

2.3 Estimating $f(d)$

We would like to associate $f(d)$ with how $E[(p_u - p_v)^2]$ varies over d, where $d = |x_u - x_v| + |y_u - y_v|$ is the Manhattan distance between pixel u and v, and p_u, p_v are the pixel values of pixel u and v respectively. $E[(p_u - p_v)^2]$ is computed over all pixel pairs u and v. Experimentally, $E[(p_u - p_v)^2]$ is found to resemble the following curve for many natural images:

$$a - \frac{b}{d + c} \tag{2.3.17}$$

where a, b and c are positive constants relative to d (see Figure 2.1 for an example and Appendix A for more plots). However, as a model for $f(d)$, this expression proves difficult to work with and we propose two approximations: piecewise approximation using 2 straight lines and piecewise approximation using a curve and a straight line. The first is elegant and greatly simplifies the analysis, but is a little coarse at smaller distances. If a more accurate approximation is required, we approximate the curve at smaller distances with a square root curve (see Figure 2.2).

2.3.1 Linear Approximation

In the following analysis, the piecewise approximation using two straight lines is as follows:

$$f(d) = \begin{cases} ud & \text{if } 0 < d \leq \alpha, \\ f_{\max} & \text{if } d > \alpha. \end{cases} \tag{2.3.18}$$

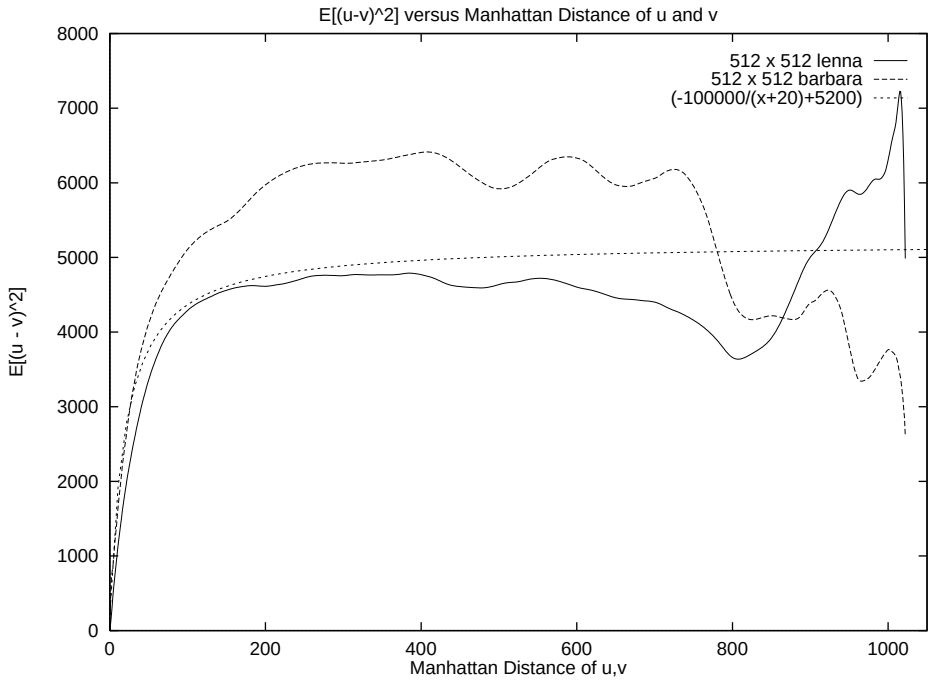


Figure 2.1: Plot of how $E[(p_u - p_v)^2]$ varies over Manhattan distance d for lenna and barbara. Values are obtained experimentally.

where u is the gradient of the first 'piece' and $f_{\max}$ represents the second 'piece' which is a horizontal line. α can be viewed as the x-coordinate value of the intersection of the two lines; hence,

$$\alpha = \frac{f_{max}}{u}. \tag{2.3.19}$$

We have set the value of u and f_{max} through experimentation.

One assumption about u requires explanation. For an image with a fixed image size of n pixels, the gradient u of $f(i)$ is constant; however, what happens when n is varied? We interpret the increase or decrease of n for a particular image to be an increase or decrease in resolution of that same image. Alternatively, this increase or decrease in resolution can also be viewed as sampling

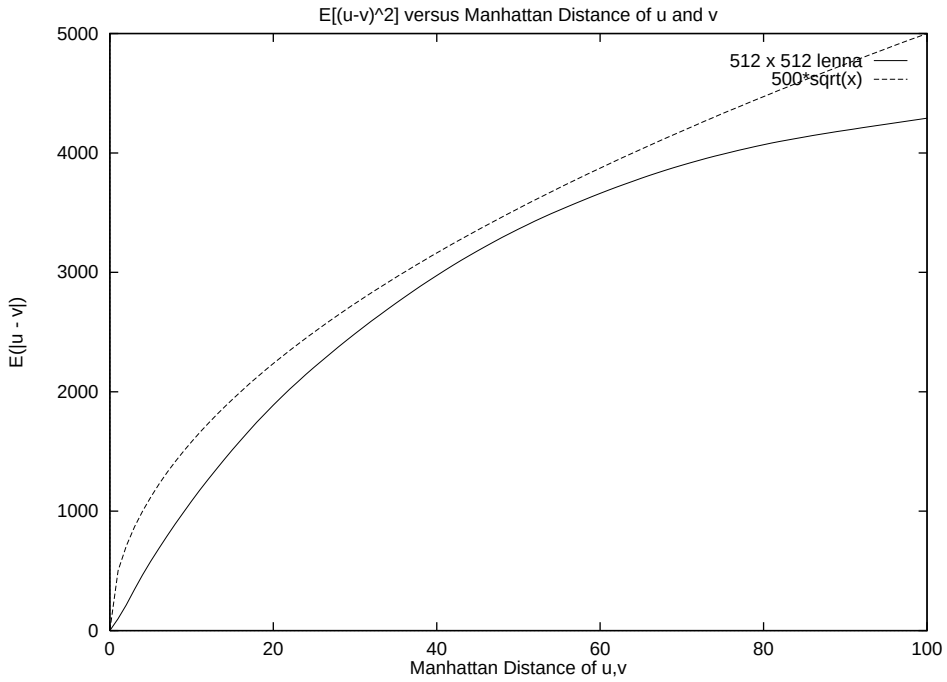


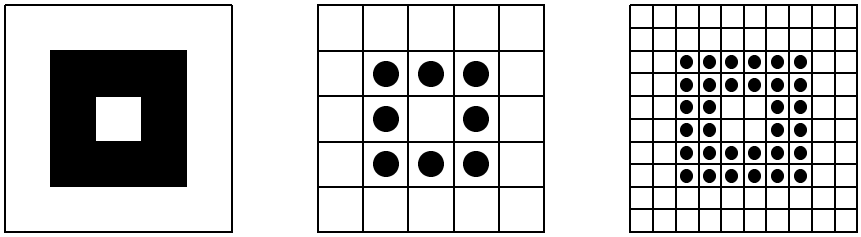
Figure 2.2: Plot of how $E[(p_u - p_v)^2]$ varies over Manhattan distance d for lenna at small distances. The square root curve $500\sqrt{d}$ is a better fit than a straight line.

the same image at more or less points. For example, consider the two bi-level images sampled at different rates from the real picture in Figure 2.3. Consider any two points, say the two lower corners of the square. The pixels corresponding these two points in the 25-pixel image and the 100-pixel image have the same pixel values, but their distances are double.

In fact, for any two points in a given image, their distance will be doubled if n is quadrupled. To account for this, we propose,

$$u = \frac{w}{\sqrt{n}} \tag{2.3.20}$$

where w is some positive constant, not dependent upon n, but dependent on



The real picture 5×5 image 10×10 image

Figure 2.3: Increasing n as an increase in resolution. The same hollow square in the real picture is digitized into a 5×5 bi-level image and a 10×10 bi-level image.

the picture. Another way to look at this is that the absolute distance between 2 points is not constant, but the relative distance ($\frac{d}{\sqrt{n}}$) is constant across all resolutions.

In most practical cases, the distance of neighbors d is seldom greater than α. For MLP, usually enough starting pixels are transmitted to make the distance of neighbors at the first level d_1 less than the threshold α. This constraint on the number of starting pixels s for MLP will be quantified as follows:

$$d_1 = \sqrt{\frac{n}{s}} \quad \text{(see Section 4.1.1)} \tag{2.3.21}$$

$$\leq \alpha. \tag{2.3.22}$$

But $\alpha = \frac{f_{max}}{u}$ (2.3.19),

$$\sqrt{\frac{n}{s}} \leq \frac{f_{max}}{u}. \tag{2.3.23}$$

And substituting $u = \frac{w}{\sqrt{n}}$ (2.3.20),

$$\sqrt{\frac{n}{s}} \leq f_{max} \div \frac{w}{\sqrt{n}} \tag{2.3.24}$$

$$\sqrt{\frac{n}{s}} \leq f_{max} \frac{\sqrt{n}}{w}. \tag{2.3.25}$$

Since all terms are positive,

$$\frac{w}{f_{max}} \leq \sqrt{s}. \tag{2.3.26}$$

Note that w and f_{max} are constants relative n and that,

$$\frac{w}{f_{max}} \leq \sqrt{s} \Rightarrow d < \alpha. \tag{2.3.27}$$

This condition effectively makes our model for $f(i)$ a straight line for MLP; hence the following analysis of MLP will assume $s = \frac{w}{f_{max}}$ and omit the flat portion.

2.3.2 Sublinear Approximation

The approximation using a curve and a straight line is defined as:

$$f(d) = \begin{cases} u\sqrt{d} & \text{if } 0 < d \leq \alpha, \\ f_{\max} & \text{if } d > \alpha. \end{cases} \tag{2.3.28}$$

where u is now a constant which both scales and dilates the curve. Note that u is constant with respect to d but dependent on n. α, the intersection point

between the two pieces as before, is now,

$$\alpha = \left(\frac{f_{max}}{u}\right)^2. \tag{2.3.29}$$

Due to the square root, the relationship between u and n is now

$$u = \frac{w}{\sqrt[4]{n}}. \tag{2.3.30}$$

and the condition for MLP to fall within the first piece of the approximation is,

$$\left(\frac{w}{f_{max}}\right)^2 \leq \sqrt{s} \Rightarrow d < \alpha. \tag{2.3.31}$$

2.4 Notation

symbol	meaning
n	total no. of pixels in the image.
s	no. of unpredictable, starting pixels.
k	refers to the k^{th} level in MLP.
$n_{k,c}$	no. of pixels predicted at level k with c neighbors.
$n_k = \sum_c n_{k,c}$	total no. of pixels predicted at level k.
d	refers to Manhattan distance in general.
d_k	Manhattan distance of neighbors at level k in MLP.
$c = \sum_d c_d$	total no. of neighbors of used in the prediction
c_d	no. of neighbors of distance d used in the prediction
m	total no. of stages to sum in a summation.
a_j, b_j, g_j, h_j	constants, indexed in order of appearance, used to simplify computations.
$\sim$	operator used in some of our approximations: $h(n) \sim g(n) \Leftrightarrow \lim_{n\to\infty} \frac{h(n)}{g(n)} = 1$ or equivalently $h(n) \sim g(n) \Leftrightarrow h(n) - g(n) = o(g(n))$.

$\mathcal{B}_{\alpha,\beta}$ usually denotes the badness of pixel ordering α using the model β for $f(d)$. For α, rs denotes the raster scan pixel ordering and for the MLP based pixel ordering, the substrings int and $last$ denote the intermediate and last levels respectively. For β, lin denotes the linear model and sub denotes the sublinear model.

The Manhattan distance of 2 points (x_u, y_u) and (x_v, y_v) is defined to be $|x_u - x_v| + |y_u - y_v|$. Unless otherwise specified, distances in this document are

Manhattan distances

The number of pixels in an image is assumed to be a perfect square and in base 2. The same conditions are required for the number of unpredictable starting pixels s in MLP.

Chapter 3

Analysis of Raster Scan

3.1 Raster Scan Pixel Ordering

The raster scan pixel ordering is a left to right, top-down scan order. Pixel choice is then constrained to those pixels in the rows above the current position and to the left of the current position if they are in the same row. Figure 3.1 shows some possible pixel choice.

3.2 Analysis of Raster Scan Pixel Ordering

In this analysis of raster scan pixel ordering, we will use all available neighbors up to a Manhattan distance of l (pixel choice **C** in Figure 3.1). Usually this implies that l strips of pixels from the left, top and right sides will need to be transmitted as unpredictable starting pixels. Hence the number of unpre-

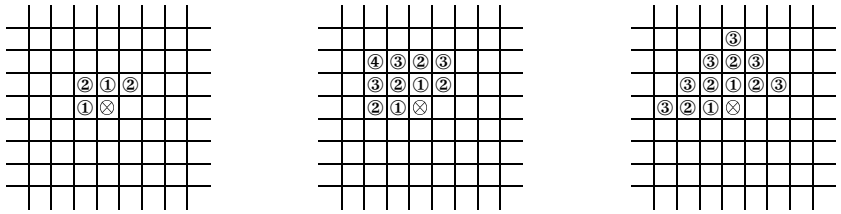


Figure 3.1: Some raster scan pixel choices. **A** and **B** are among the most popularly used. The crossed circle ⊗ represents the pixel to be predicted and the circles with a number in it denote the neighbors and their corresponding distance. Most of our analysis will use the pyramid shape pixel choice in **C**.

dictable starting pixels,

$$s = (3l)\sqrt{n} - 2l^2 \tag{3.2.1}$$

where n is the number of pixels in the image. However, since z', the cost of encoding unpredicted pixels, is a constant, it will be shown that sz' is a lower order term and therefore not significant.

Applying the new badness measure, the badness of raster scan pixel ordering is,

$$\mathcal{B}_{rs} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l} c_d f(d)}}{\sum_{d=1}^{l} c_d} \tag{3.2.2}$$

where c_d is the number of neighbors at distance d. For rasterscan, $c_d \leq 2d$ (see Figure 3.1); using the maximum c_d,

$$\mathcal{B}_{rs} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l} (2d) f(d)}}{\sum_{d=1}^{l} 2d}. \tag{3.2.3}$$

3.2.1 Analysis Using Linear Model for $f(d)$

Substituting $f(d) = ud$ (2.3.18),

$$\mathcal{B}_{rs,lin} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l}(2d)(ud)}}{2\sum_{d=1}^{l} d} \tag{3.2.4}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\sum_{d=1}^{l} d^2}}{2\sum_{d=1}^{l} d} \tag{3.2.5}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\sum_{d=1}^{l} d^2}}{\sqrt{2}\sum_{d=1}^{l} d}. \tag{3.2.6}$$

Using the identities $\sum_{d=1}^{l} d^2 = \frac{l}{6}(l+1)(2l+1)$ and $\sum_{d=1}^{l} d = \frac{l}{2}(l+1)$,

$$\mathcal{B}_{rs,lin} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\left[\frac{l}{6}(l+1)(2l+1)\right]}}{\sqrt{2}\left[\frac{l}{2}(l+1)\right]} \tag{3.2.7}$$

$$= sz' + \sum_{p=1}^{n-s} \sqrt{\frac{u(2l+1)}{3l(l+1)}} \tag{3.2.8}$$

$$= sz' + (n-s)\sqrt{\frac{u(2l+1)}{3l(l+1)}}. \tag{3.2.9}$$

Substituting $u = \frac{w}{\sqrt{n}}$ (2.3.20)

$$\mathcal{B}_{rs,lin} = sz' + (n-s)\sqrt{\frac{w}{\sqrt{n}}\left[\frac{(2l+1)}{3l(l+1)}\right]} \tag{3.2.10}$$

$$= sz' + \left(n^{\frac{3}{4}} - sn^{-\frac{1}{4}}\right)\sqrt{\frac{w(2l+1)}{3l(l+1)}}.. \tag{3.2.11}$$

3.2.2 Analysis Using Sublinear Model for $f(d)$

From Equation 3.2.3,

$$\mathcal{B}_{rs,sub} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l}(2d)f(d)}}{\sum_{d=1}^{l} 2d}. \tag{3.2.12}$$

Substituting $f(d) = u\sqrt{d}$ (2.3.28),

$$\mathcal{B}_{rs,sub} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l}(2d)(u\sqrt{d})}}{2\sum_{d=1}^{l} d} \tag{3.2.13}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{2\sum_{d=1}^{l} d} \tag{3.2.14}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{\sqrt{2}\sum_{d=1}^{l} d}. \tag{3.2.15}$$

Using the identity $\sum_{d=1}^{l} d = \frac{l}{2}(l+1)$,

$$\mathcal{B}_{rs,sub} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{\sqrt{2}\left[\frac{l}{2}(l+1)\right]} \tag{3.2.16}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{l(l+1)}. \tag{3.2.17}$$

Using the approximation $\sum_{i=1}^{l} i^{\frac{3}{2}} \approx \int_{0.5}^{l+0.5} i^{\frac{3}{2}} di = \frac{2}{5}(l+0.5)^{\frac{5}{2}} - \frac{2}{5}(0.5)^{\frac{5}{2}}$,

$$\mathcal{B}_{rs,sub} \sim sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\left[\frac{2}{5}(l+0.5)^{\frac{5}{2}} - \frac{2}{5}(0.5)^{\frac{5}{2}}\right]}}{l(l+1)}. \tag{3.2.18}$$

Note that maximum distance of a neighbor l is assumed to be much smaller than n; hence the approximation does not change $\mathcal{B}_{rs,sub}$ asymptotically in n. Next, we substitute $u = \frac{w}{\sqrt[4]{n}}$ (2.3.30)

$$\mathcal{B}_{rs,sub} \sim sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2\frac{w}{\sqrt[4]{n}}\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)} \tag{3.2.19}$$

$$= sz' + (n-s)\frac{n^{-\frac{1}{8}}\sqrt{2w\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)}. \tag{3.2.20}$$

Finally,

$$\mathcal{B}_{rs,sub} \sim \left\{\frac{\sqrt{2w\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)}\right\} n^{\frac{7}{8}} + sz'$$
$$- \left\{\frac{\sqrt{2w\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)}\right\} n^{-\frac{1}{8}}. \tag{3.2.21}$$

3.3 The Badness of Raster Scan Pixel Ordering

The badness of raster scan pixel ordering according to our two models for $f(d)$ is summarized in Table 3.1 for $1 \leq l \leq 7$. Only the dominant term is shown since the entire expression is complicated.

Having analyzed raster scan, we proceed with the analysis of the MLP pixel ordering in the next chapter to see how MLP perform against raster scan.

l	linear model	sublinear model
1	$0.707\sqrt{w}n^{\frac{3}{4}}$	$(0.718 + O(1))\sqrt{w}n^{\frac{7}{8}}$
2	$0.527\sqrt{w}n^{\frac{3}{4}}$	$(0.464 + O(1))\sqrt{w}n^{\frac{7}{8}}$
3	$0.441\sqrt{w}n^{\frac{3}{4}}$	$(0.355 + O(1))\sqrt{w}n^{\frac{7}{8}}$
4	$0.387\sqrt{w}n^{\frac{3}{4}}$	$(0.293 + O(1))\sqrt{w}n^{\frac{7}{8}}$
5	$0.350\sqrt{w}n^{\frac{3}{4}}$	$(0.251 + O(1))\sqrt{w}n^{\frac{7}{8}}$
6	$0.321\sqrt{w}n^{\frac{3}{4}}$	$(0.221 + O(1))\sqrt{w}n^{\frac{7}{8}}$
7	$0.300\sqrt{w}n^{\frac{3}{4}}$	$(0.198 + O(1))\sqrt{w}n^{\frac{7}{8}}$

Table 3.1: Summary of the badness of raster scan pixel ordering. Only the dominant term is shown.

Chapter 4

Analysis of the Multi Level Progressive (MLP) Pixel Ordering

In this chapter, we introduce the MLP pixel ordering as described in [HV92b] and show how to analyze it with our badness measure. We then generalize the original MLP method to use pixel choices that include more neighbors in the hope of a better pixel ordering and also a fairer comparison with raster scan orderings. Two particular case of this generalized MLP which we call GMLP6 and GMLP18 are analyzed.

4.1 Analysis of MLP

4.1.1 The MLP algorithm

The Multi-Level Progressive method (MLP) consists of three main steps: prediction, variance estimation and encoding of prediction error using the estimated variance with a Laplacian distribution. The second and third steps are not directly related to the pixel ordering problem and will not be described here. The reader should consult [HV92b] and [HV92a] if interested. Before applying our new measure to analyze MLP's pixel ordering, we will describe briefly the MLP prediction step and formulate some of its relevant aspects mathematically .

The MLP prediction step proceeds in levels. At each level, pixels are predicted using some number of neighboring pixels which have been predicted in the previous level. The last level uses a pixel choice of 16 neighbors which form a diamond shape around the predicted pixel. The other levels use a pixel choice consisting of the 4 nearest lattice pixels which form either a square or a 45°-tilted square around the predicted pixel. The predicted pixel value is typically a weighted sum of its neighbors' values. Prediction of pixels near the edges of the image will typically have some neighbors missing, since these neighbors occur theoretically beyond the edges. In such cases, the weight of the missing neighbor is set to zero and the non-zero weights renormalized. Effectively this means that less neighbors are used for prediction.

The number of pixels predicted doubles after each level; therefore, the

number of pixels predicted at level k can be formulated recursively as,

$$n_k = \begin{cases} s & \text{if } k = 1 \\ 2n_{k-1} & \text{otherwise.} \end{cases} \tag{4.1.1}$$

$$= 2^{k-1}s. \tag{4.1.2}$$

Using this expression and the fact that at the last level the number of pixels predicted must be $\frac{n}{2}$, we can find the total number of levels required to process an n-pixel image as follows,

$$\begin{aligned} n_{last} &= \frac{n}{2} \\ 2^{last-1}s &= \frac{n}{2} \\ 2^{last} &= \frac{n}{s} \\ last &= \log_2\left(\frac{n}{s}\right). \end{aligned} \tag{4.1.3}$$

Since n and s are assumed to be perfect squares in base 2, $last$ is even.

Except the last level each prediction uses the 4 nearest lattice pixels that have been predicted in the previous level. The distance of the neighbors used for prediction in all except the last level reduces by a factor of two after every 2 levels (see Figure 4.2). Since n and s are assumed to be perfect squares and in base 2, the distance of the neighbors at the start of the first level can be expressed as,

$$d_1 = \sqrt{\frac{n}{s}} \tag{4.1.4}$$

and the distance of the neighbors used in the prediction at level k (except last level) as,

$$d_k = \begin{cases} \sqrt{\frac{n}{s}} & \text{if } k = 1, \\ d_{k-1} & \text{if } k \text{ is odd} \\ \frac{1}{2}d_{k-1} & \text{if } k \text{ is even} \end{cases} \tag{4.1.5}$$

$$= 2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}. \tag{4.1.6}$$

Each pixel of the image is either transmitted as an unpredictable starting pixel or predicted exactly once at exactly one level; therefore, we can partition these pixels according to the level at which they are predicted. At each level,

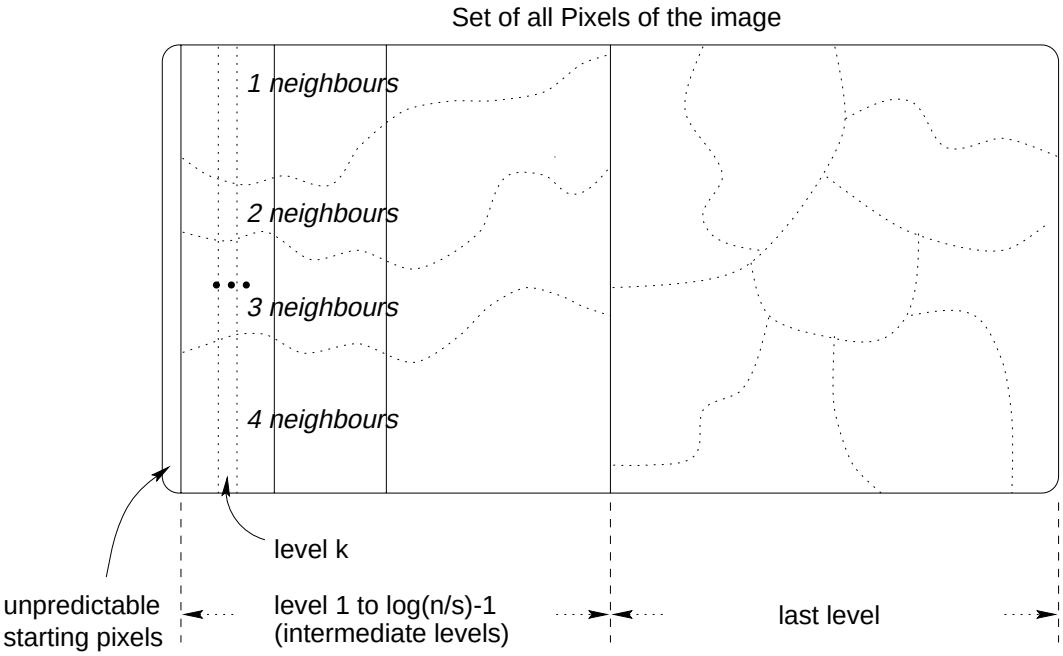


Figure 4.1: How the pixels are partitioned and summed for MLP.

the pixels can further be partitioned into those which are predicted using one,

two, three or four neighbors for the intermediate levels and up to sixteen neighbors for the last level. The prediction using less than the specified four and sixteen neighbors are those for pixels occurring near the edges of the image. We use this two tier partition scheme to decompose the summation in our badness measure (see Figure 4.1),

$$\begin{aligned}\mathcal{B}_{MLP} &= sz' + \mathcal{B}_{MLP-intermediate} + \mathcal{B}_{MLP-last} \\ &= sz' + \sum_{k=1}^{m}\left\{\sum_{c=1}^{4} n_{k,c}\frac{\sqrt{\sum_d c_d f(d)}}{c}\right\} + \sum_{c=1}^{16} n_{last,c}\frac{\sqrt{\sum_d c_d f(d)}}{c} \quad (4.1.7)\end{aligned}$$

where m is the number of intermediate levels and $n_{k,c}$ the number of pixels in the partition representing predictions in level k and with c neighbors. The expressions for $n_{k,c}$ at the intermediate and final levels are tabulated in Table 4.1 and Table 4.2 respectively. By applying these expressions to the example in Figure 4.2, the reader can have some idea of their derivation.

The second and third terms are worked out in section 4.1.2 using $f(d) = ud$ and in section 4.1.3 using $f(d) = u\sqrt{d}$.

4.1.2 Analysis Using Linear Model for $f(d)$

The Analysis of the Intermediate Levels using Linear Model for $f(d)$

We note that at the intermediate levels, all the neighbors used for prediction at a particular level k are at a fixed distance d_k from the predicted pixel.

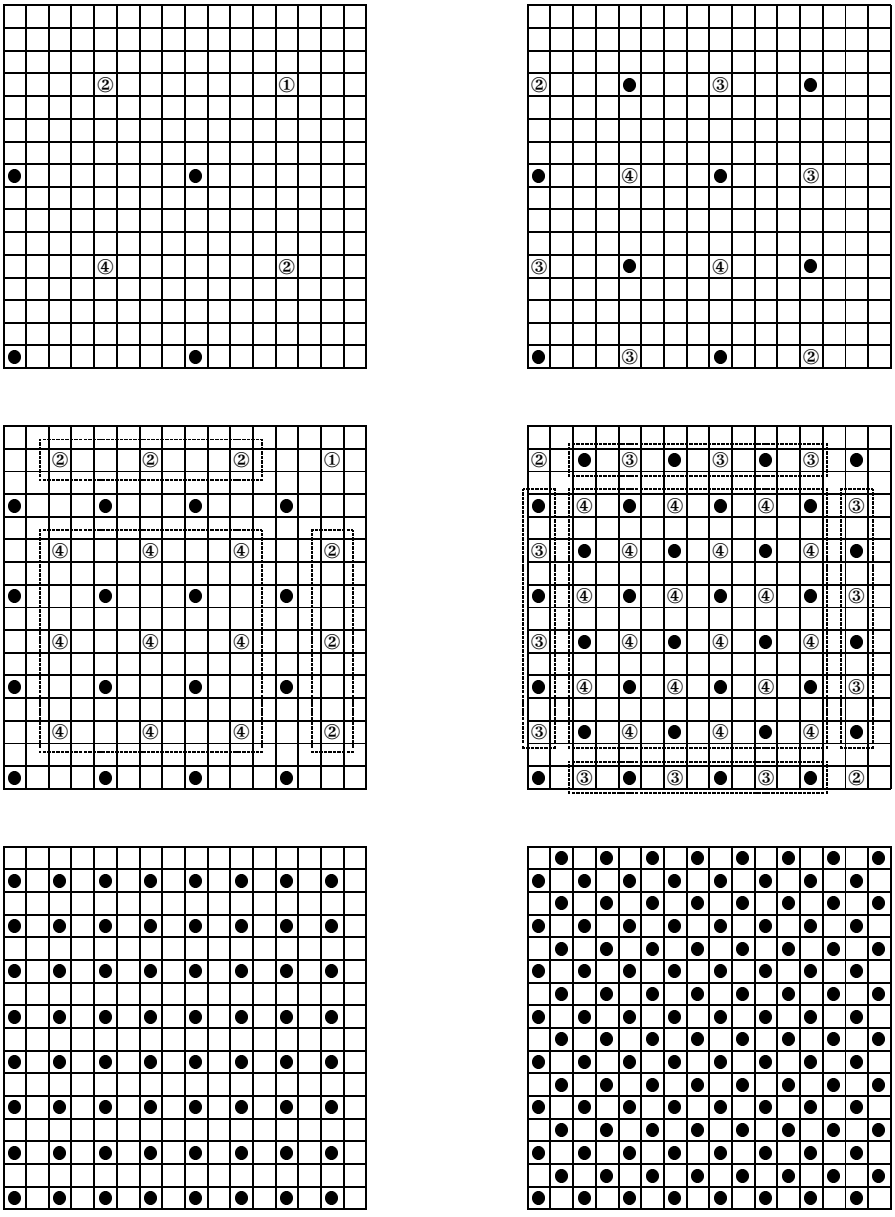


Figure 4.2: Some MLP levels of a 16 by 16 image. Odd levels are on the left column and even levels on the right. The empty circles ∘ denotes pixels predicted in previous levels and the circles with a number in it denote the pixels predicted in the current level with the number denoting the number of neighbors used to predict that pixel. By observing the pattern of these numbered circles, the expressions for $n_{k,c}$ can be derived. The numbered circles are omitted from the last level for clarity.

c	$n_{k,c}$ for odd k	$n_{k,c}$ for even k
1	1	0
2	$2(\sqrt{n_k}-1)$	2
3	0	$4(\sqrt{n_{k-1}}-1)$
4	$(\sqrt{n_k}-1)^2$	$(\sqrt{n_{k-1}}-1)(2\sqrt{n_{k-1}}-2)$

Table 4.1: $n_{k,c}$ for MLP intermediate levels.

Summing from an arbitrary level v to m,

$$\mathcal{B}_{MLP-int,lin} = \sum_{k=v}^{m}\left[\sum_{c=1}^{4} n_{k,c}\frac{\sqrt{cf(d_k)}}{c}\right] \tag{4.1.8}$$

$$= \sum_{k=v}^{m}\left[\sum_{c=1}^{4} n_{k,c}\sqrt{\frac{f(d_k)}{c}}\right] \tag{4.1.9}$$

$$= \sum_{k=v}^{m}\left[n_{k,1}\sqrt{\frac{f(d_k)}{1}} + n_{k,2}\sqrt{\frac{f(d_k)}{2}} + n_{k,3}\sqrt{\frac{f(d_k)}{3}} + n_{k,4}\sqrt{\frac{f(d_k)}{4}}\right]. \tag{4.1.10}$$

Splitting the summation into odd and even k

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} = &\sum_{\substack{k=v\\ k \text{ odd}}}^{m}\sqrt{f(d_k)}\left[n_{k,1} + \frac{1}{\sqrt{2}}n_{k,2} + \frac{1}{\sqrt{3}}n_{k,3} + \frac{1}{2}n_{k,4}\right]\\ &+ \sum_{\substack{k=v\\ k \text{ even}}}^{m}\sqrt{f(d_k)}\left[n_{k,1} + \frac{1}{\sqrt{2}}n_{k,2} + \frac{1}{\sqrt{3}}n_{k,3} + \frac{1}{2}n_{k,4}\right].\end{aligned} \tag{4.1.11}$$

Substituting the expressions for $n_{k,c}$ from Table 4.1,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} = &\sum_{\substack{k=v\\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \left[1 + \frac{2}{\sqrt{2}}\left(\sqrt{n_k} - 1\right) + \frac{1}{2}\left(\sqrt{n_k} - 1\right)^2\right] \\ &+ \sum_{\substack{k=v\\ k \text{ even}}}^{m} \sqrt{f(d_k)}\left[\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{3}}\left(\sqrt{n_{k-1}} - 1\right)\right. \\ &\qquad \left. + \frac{1}{2}\left(\sqrt{n_{k-1}} - 1\right)\left(2\sqrt{n_{k-1}} - 2\right)\right] \end{aligned} \tag{4.1.12}$$

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} = &\sum_{\substack{k=v\\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \left[1 + \sqrt{2}\left(\sqrt{n_k}\right) - \sqrt{2} + \frac{1}{2}\left(n_k - 2\sqrt{n_k} + 1\right)\right] \\ &+ \sum_{\substack{k=v\\ k \text{ even}}}^{m} \sqrt{f(d_k)}\left[\sqrt{2} + \frac{4}{\sqrt{3}}\left(\sqrt{n_{k-1}}\right) - \frac{4}{\sqrt{3}}\right. \\ &\qquad \left. + \left(n_{k-1} - 2\sqrt{n_{k-1}} + 1\right)\right] \end{aligned} \tag{4.1.13}$$

$$\begin{aligned}= &\sum_{\substack{k=v\\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \left[\frac{1}{2}n_k + \left(\sqrt{2} - 1\right)\sqrt{n_k} + \left(\frac{3}{2} - \sqrt{2}\right)\right] \\ &+ \sum_{\substack{k=v\\ k \text{ even}}}^{m} \sqrt{f(d_k)}\left[n_{k-1} + \left(\frac{4}{\sqrt{3}} - 2\right)\sqrt{n_{k-1}} + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right)\right]. \end{aligned} \tag{4.1.14}$$

Substituting $n_k = 2^{k-1}s$ (4.1.2)

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} &= \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \left[\frac{1}{2}2^{k-1}s + \left(\sqrt{2}-1\right)\sqrt{2^{k-1}s} + \left(\frac{3}{2}-\sqrt{2}\right)\right] \\ &\quad + \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{f(d_k)} \left[2^{k-2}s + \left(\frac{4}{\sqrt{3}}-2\right)\sqrt{2^{k-2}s} + \left(\sqrt{2}-\frac{4}{\sqrt{3}}+1\right)\right] \qquad (4.1.15) \\ &= \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \left[\frac{1}{4}2^{k}s + \left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{k/2} + \left(\frac{3}{2}-\sqrt{2}\right)\right] \\ &\quad + \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{f(d_k)} \left[\frac{1}{4}2^{k}s + \left(\frac{2}{\sqrt{3}}-1\right)\sqrt{s}2^{k/2} + \left(\sqrt{2}-\frac{4}{\sqrt{3}}+1\right)\right]. \qquad (4.1.16)\end{aligned}$$

Substituting $f(d_k) = ud_k$ (2.3.18) and $d_k = 2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}$ (4.1.6),

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} &= \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{u2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}} \left[\frac{1}{4}2^{k}s + \left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{k/2} + \left(\frac{3}{2}-\sqrt{2}\right)\right] \\ &\quad + \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{u2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}} \left[\frac{1}{4}2^{k}s + \left(\frac{2}{\sqrt{3}}-1\right)\sqrt{s}2^{k/2} \right. \\ &\qquad \left. + \left(\sqrt{2}-\frac{4}{\sqrt{3}}+1\right)\right]. \qquad (4.1.17)\end{aligned}$$

If k is even, $\lfloor k/2 \rfloor = k/2$, else $\lfloor k/2 \rfloor = (k-1)/2$,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} = &\sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{u2^{-(k-1)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^k s + \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{s}2^{k/2} + \left(\frac{3}{2} - \sqrt{2}\right)\right] \\ &+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{u2^{-k/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^k s + \left(\frac{2}{\sqrt{3}} - 1\right)\sqrt{s}2^{k/2}\right. \\ &\qquad \left. + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right)\right]. \qquad (4.1.18)\end{aligned}$$

Since $last$ is even, $m = last - 1$ is odd, but we want the two summations to be over the same number of levels. Hence we bring out the term when $k = m$ from the summation over odd k's to make the number of odd levels the same as the number of even levels in the 2 summations,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} = &\left\{\sum_{\substack{k=v \\ k \text{ odd}}}^{m-1} \sqrt{u2^{-(k-1)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^k s + \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{s}2^{k/2} + \left(\frac{3}{2} - \sqrt{2}\right)\right]\right. \\ &\left. + \sqrt{u2^{-(m-1)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^m s + \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{s}2^{m/2} + \left(\frac{3}{2} - \sqrt{2}\right)\right]\right\} \\ &+ \sum_{\substack{k=v \\ k \text{ even}}}^{m-1} \sqrt{u2^{-k/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^k s + \left(\frac{2}{\sqrt{3}} - 1\right)\sqrt{s}2^{k/2}\right. \\ &\qquad \left. + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right)\right]. \qquad (4.1.19)\end{aligned}$$

Change indexing variable $k \to j$, where odd $k = v + 2j - 2$ and even $k = v + 2j - 1$, assuming v is odd,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} = &\sum_{j=1}^{(m-v)/2} \sqrt{u2^{-\frac{v+2j-3}{2}}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^{v+2j-2}s + \left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{\frac{v+2j-2}{2}} \right.\\ &\left. + \left(\frac{3}{2}-\sqrt{2}\right)\right]\\ &+ \sum_{j=1}^{(m-v)/2} \sqrt{u2^{-\frac{v+2j-1}{2}}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^{v+2j-1}s + \left(\frac{2}{\sqrt{3}}-1\right)\sqrt{s}2^{\frac{v+2j-1}{2}} \right.\\ &\left. + \left(\sqrt{2}-\frac{4}{\sqrt{3}}+1\right)\right]\\ &+ \sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^{m}s + \left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{m/2} + \left(\frac{3}{2}-\sqrt{2}\right)\right].\end{aligned} \tag{4.1.20}$$

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} = &\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\sum_{j=1}^{(m-v)/2}\left\{2^{\frac{1-j}{2}}\left[\frac{1}{4}2^{v+2j-2}s + \left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{\frac{v+2j-2}{2}} \right.\right.\\ &\left. + \left(\frac{3}{2}-\sqrt{2}\right)\right]\\ &+ 2^{-\frac{j}{2}}\left[\frac{1}{4}2^{v+2j-1}s + \left(\frac{2}{\sqrt{3}}-1\right)\sqrt{s}2^{\frac{v+2j-1}{2}} \right.\\ &\left.\left. + \left(\sqrt{2}-\frac{4}{\sqrt{3}}+1\right)\right]\right\}\\ &+ \sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^{m}s + \left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{m/2} + \left(\frac{3}{2}-\sqrt{2}\right)\right]\end{aligned} \tag{4.1.21}$$

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\sum_{j=1}^{(m-v)/2}\Bigg\{\Bigg[\frac{1}{4}2^{v+\frac{3}{2}j-\frac{3}{2}}s+\left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{\frac{v+j-1}{2}}\\&\qquad+\left(\frac{3}{\sqrt{2}}-2\right)2^{-\frac{j}{2}}\Bigg]\\&\qquad+\Bigg[\frac{1}{4}2^{v+\frac{3}{2}j-1}s+\left(\frac{2}{\sqrt{3}}-1\right)\sqrt{s}2^{\frac{v+j-1}{2}}\\&\qquad\quad+\left(\sqrt{2}-\frac{4}{\sqrt{3}}+1\right)2^{-\frac{j}{2}}\Bigg]\Bigg\}\\&+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^{m}s+\left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{m/2}+\left(\frac{3}{2}-\sqrt{2}\right)\right].\end{aligned}\tag{4.1.22}$$

Collecting j-terms,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\sum_{j=1}^{(m-v)/2}\Bigg\{\left[\frac{1}{4\sqrt{8}}+\frac{1}{8}\right]s2^{v}2^{\frac{3}{2}j}\\&\qquad+\left[\frac{1}{\sqrt{2}}\left(1-\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{3}}-1\right)\right]\sqrt{s}2^{\frac{v}{2}}2^{\frac{j}{2}}\\&\qquad+\left[\left(\frac{3}{\sqrt{2}}-2\right)+\left(\sqrt{2}-\frac{4}{\sqrt{3}}+1\right)\right]2^{-\frac{j}{2}}\Bigg\}\\&+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^{m}s+\left(1-\frac{1}{\sqrt{2}}\right)\sqrt{s}2^{m/2}+\left(\frac{3}{2}-\sqrt{2}\right)\right]\end{aligned}\tag{4.1.23}$$

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\sum_{j=1}^{(m-v)/2}\left\{\left[\frac{1}{4\sqrt{8}}+\frac{1}{8}\right]s2^v2^{\frac{3}{2}j}\right.\\ &\qquad+\left[\sqrt{\frac{2}{3}}-\frac{1}{2}\right]\sqrt{s}2^{\frac{v}{2}}2^{\frac{j}{2}}\\ &\qquad\left.+\left[\frac{5}{\sqrt{2}}-\frac{4}{\sqrt{3}}-1\right]2^{-\frac{j}{2}}\right\}\\ &+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left\{\frac{1}{4}2^ms+\left[1-\frac{1}{\sqrt{2}}\right]\sqrt{s}2^{m/2}+\left[\frac{3}{2}-\sqrt{2}\right]\right\}.\end{aligned} \tag{4.1.24}$$

Replacing the constant expressions in square brackets with a_1, $a_2 \ldots a_5$ respectively,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\sum_{j=1}^{\frac{m-v}{2}}\left\{a_1s2^v2^{\frac{3}{2}j}+a_2\sqrt{s}2^{\frac{v}{2}}2^{\frac{j}{2}}+a_32^{-\frac{j}{2}}\right\}\\ &+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^ms+a_4\sqrt{s}2^{m/2}+a_5\right] \qquad (4.1.25)\\ =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\left\{a_1s2^v\sum_{j=1}^{\frac{m-v}{2}}(2^{\frac{3}{2}j})+a_2\sqrt{s}2^{\frac{v}{2}}\sum_{j=1}^{\frac{m-v}{2}}(2^{\frac{j}{2}})+a_3\sum_{j=1}^{\frac{m-v}{2}}(2^{-\frac{j}{2}})\right\}\\ &+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^ms+a_4\sqrt{s}2^{\frac{m}{2}}+a_5\right]. \qquad (4.1.26)\end{aligned}$$

Now, the summations are just geometric series,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\left\{ \quad a_1 s2^v 2^{\frac{3}{2}}\left(\frac{2^{\frac{3}{2}(\frac{m-v}{2})}-1}{2^{\frac{3}{2}}-1}\right)\right.\\ &\qquad +a_2\sqrt{s}2^{\frac{v}{2}}2^{\frac{1}{2}}\left(\frac{2^{\frac{1}{2}(\frac{m-v}{2})}-1}{2^{\frac{1}{2}}-1}\right)\\ &\qquad \left.+a_3 2^{-\frac{1}{2}}\left(\frac{1-2^{-\frac{1}{2}(\frac{m-v}{2})}}{1-2^{-\frac{1}{2}}}\right)\right\}\\ &+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^m s+a_4\sqrt{s}2^{\frac{m}{2}}+a_5\right] \qquad (4.1.27)\\ =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\left\{ \quad s2^v\left[\frac{a_1 2^{\frac{3}{2}}}{2^{\frac{3}{2}}-1}\right]\left(2^{\frac{3}{4}(m-v)}-1\right)\right.\\ &\qquad +\sqrt{s}2^{\frac{v}{2}}\left[\frac{a_2 2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1}\right]\left(2^{\frac{1}{4}(m-v)}-1\right)\\ &\qquad \left.+\left[\frac{a_3 2^{-\frac{1}{2}}}{1-2^{-\frac{1}{2}}}\right]\left(1-2^{-\frac{1}{4}(m-v)}\right)\right\}\\ &+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^m s+a_4\sqrt{s}2^{\frac{m}{2}}+a_5\right]. \qquad (4.1.28)\end{aligned}$$

Replacing constant expressions in $[\ldots]$ again,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\left\{ \quad s2^v a_6\left(2^{\frac{3}{4}(m-v)}-1\right)\right.\\ &\qquad +\sqrt{s}2^{\frac{v}{2}}a_7\left(2^{\frac{1}{4}(m-v)}-1\right)\\ &\qquad \left.+a_8\left(1-2^{-\frac{1}{4}(m-v)}\right)\right\}\\ &+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^m s+a_4\sqrt{s}2^{\frac{m}{2}}+a_5\right] \qquad (4.1.29)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\Big[\quad\left(sa_6 2^{\frac{1}{4}v}\right)2^{\frac{3}{4}m} - s2^v a_6\\ &\qquad+\left(\sqrt{s}a_7 2^{\frac{1}{4}v}\right)2^{\frac{1}{4}m} - \sqrt{s}2^{\frac{v}{2}}a_7\\ &\qquad+a_8 - \left(a_8 2^{\frac{1}{4}v}\right)2^{-\frac{1}{4}m}\Big]\\ &+\sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^m s + a_4\sqrt{s}2^{\frac{m}{2}} + a_5\right]. \end{aligned} \tag{4.1.30}$$

Substituting $m = \log_2\left(\frac{n}{s}\right) - 1$ and consolidating non-m-terms at the end,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =& \sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}}\Bigg[\quad \left(sa_6 2^{\frac{1}{4}v}\right) 2^{\frac{3}{4}[\log_2\left(\frac{n}{s}\right)-1]} \\ &+ \left(\sqrt{s}a_7 2^{\frac{1}{4}v}\right) 2^{\frac{1}{4}[\log_2\left(\frac{n}{s}\right)-1]} \\ &- \left(a_8 2^{\frac{1}{4}v}\right) 2^{-\frac{1}{4}[\log_2\left(\frac{n}{s}\right)-1]} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^v a_6\right)\Bigg] \\ &+ \sqrt{u2^{[1-\log_2\left(\frac{n}{s}\right)+1]/2}\sqrt{\frac{n}{s}}}\left[\frac{1}{4}2^{\log_2\left(\frac{n}{s}\right)-1}s + a_4\sqrt{s}2^{\frac{1}{2}[\log_2\left(\frac{n}{s}\right)-1]} + a_5\right]\end{aligned} \tag{4.1.31}$$

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =& \sqrt{u}2^{\frac{1-v}{4}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\Bigg[\quad \left(sa_6 2^{\frac{1}{4}v}\right) 2^{-\frac{3}{4}}\left(\frac{n}{s}\right)^{\frac{3}{4}} \\ &+ \left(\sqrt{s}a_7 2^{\frac{1}{4}v}\right) 2^{-\frac{1}{4}}\left(\frac{n}{s}\right)^{\frac{1}{4}} \\ &- \left(a_8 2^{\frac{1}{4}v}\right) 2^{\frac{1}{4}}\left(\frac{n}{s}\right)^{-\frac{1}{4}} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^v a_6\right)\Bigg] \\ &+ \sqrt{u}\left(\frac{n}{s}\right)^{-\frac{1}{4}} 2^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\left[\frac{1}{8}\left(\frac{n}{s}\right)s + a_4\sqrt{s}2^{-\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{2}} + a_5\right].\end{aligned} \tag{4.1.32}$$

Distributing $2^{\frac{1-v}{4}}\left(\frac{n}{s}\right)^{\frac{1}{4}}$ into $[\ldots]$ for the first term and simplifying the second,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u}\Bigg[\quad\left(2^{-\frac{1}{2}}sa_6\right)\left(\frac{n}{s}\right)\\&+\left(\sqrt{s}a_7\right)\left(\frac{n}{s}\right)^{\frac{1}{2}}\\&-\left(a_8 2^{\frac{1}{2}}\right)\\&+\left(a_8-\sqrt{s}2^{\frac{v}{2}}a_7-s2^v a_6\right)2^{\frac{1-v}{4}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\Bigg]\\&+\sqrt{u}2^{\frac{1}{2}}\left[\frac{1}{8}n+2^{-\frac{1}{2}}a_4 n^{\frac{1}{2}}+a_5\right].\end{aligned}\tag{4.1.33}$$

Finally,

$$\begin{aligned}\mathcal{B}_{MLP-int,lin} =&\sqrt{u}\Bigg\{\quad\left(2^{-\frac{1}{2}}a_6+\frac{\sqrt{2}}{8}\right)n+\left(a_7+a_4\right)n^{\frac{1}{2}}\\&+\left(a_8-\sqrt{s}2^{\frac{v}{2}}a_7-s2^v a_6\right)\left(2^{\frac{1-v}{4}}s^{-\frac{1}{4}}\right)n^{\frac{1}{4}}\\&+\left(2^{\frac{1}{2}}a_5-2^{\frac{1}{2}}a_8\right)\Bigg\}\end{aligned}\tag{4.1.34}$$

where constants $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and a_8 are:

$$a_1 = \frac{1}{4\sqrt{8}} + \frac{1}{8} \tag{4.1.35}$$

$$a_2 = \sqrt{\frac{2}{3}} - \frac{1}{2} \tag{4.1.36}$$

$$a_3 = \frac{5}{\sqrt{2}} - \frac{4}{\sqrt{3}} - 1 \tag{4.1.37}$$

$$a_4 = 1 - \frac{1}{\sqrt{2}} \qquad \approx 0.2923 \tag{4.1.38}$$

$$a_5 = \frac{3}{2} - \sqrt{2} \qquad \approx 0.0858 \tag{4.1.39}$$

$$a_6 = \frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}} - 1} \left(\frac{1}{4\sqrt{8}} + \frac{1}{8} \right) \qquad \approx 0.3301 \tag{4.1.40}$$

$$a_7 = \frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}} - 1} \left(\sqrt{\frac{2}{3}} - \frac{1}{2} \right) \qquad \approx 1.0806 \tag{4.1.41}$$

$$a_8 = \frac{2^{-\frac{1}{2}}}{1 - 2^{-\frac{1}{2}}} \left(\frac{5}{\sqrt{2}} - \frac{4}{\sqrt{3}} - 1 \right) \qquad \approx 0.5459. \tag{4.1.42}$$

The Analysis of the Last Level using Linear Model for $f(d)$

$$\mathcal{B}_{MLP-last,lin} = \sum_{c=1}^{16} n_{last,c} \frac{\sqrt{\sum_d c_d f(d)}}{c}. \tag{4.1.43}$$

Substituting $f(d) = ud$ (2.3.18),

$$\mathcal{B}_{MLP-last,lin} = \sqrt{u} \sum_{c=1}^{16} n_{last,c} \frac{\sqrt{\sum_d c_d d}}{c}. \tag{4.1.44}$$

$c = \sum_d c_d$	c_1	c_3	$n_{k,c}$ for even last level k
6	2	4	2
8	3	5	4
9	3	6	4
10	3	7	$2 + 4\left(\frac{\sqrt{n}-6}{2}\right)$
12	4	8	4
13	4	9	$4\left(\frac{\sqrt{n}-6}{2}\right)$
14	4	10	2
15	4	11	$4\left(\frac{\sqrt{n}-6}{2}\right)$
16	4	12	$(\sqrt{n}-6)\left(\frac{\sqrt{n}-6}{2}\right)$
otherwise			0

Table 4.2: $n_{k,c}$ for MLP last level. Our assumptions on n and s will make the last level even.

Substituting the expressions for $n_{k,c}$ from Table 4.2,

$$\begin{aligned}\mathcal{B}_{MLP-last,lin} = \sqrt{u}\Bigg\{ & 2 \times \frac{\sqrt{2+4\times 3}}{6} + 4 \times \frac{\sqrt{3+5\times 3}}{8} + 4 \times \frac{\sqrt{3+6\times 3}}{9} \\ & + \left[2 + 4\left(\frac{\sqrt{n}-6}{2}\right)\right]\frac{\sqrt{3+7\times 3}}{10} + 4 \times \frac{\sqrt{4+8\times 3}}{12} \\ & + 4\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{4+9\times 3}}{13} + 2 \times \frac{\sqrt{4+10\times 3}}{14} \\ & + 4\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{4+11\times 3}}{15} \\ & + \left(\sqrt{n}-6\right)\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{4+12\times 3}}{16}\Bigg\}. \end{aligned} \tag{4.1.45}$$

Simplifying square roots,

$$\begin{aligned}\mathcal{B}_{MLP-last,lin} = \sqrt{u}\Bigg\{ & \frac{\sqrt{14}}{3} + \frac{3}{2}\sqrt{2} + \frac{4}{9}\sqrt{21} \\ & + \left[2 + 4\left(\frac{\sqrt{n}-6}{2}\right)\right]\frac{\sqrt{6}}{5} + \frac{2}{3}\sqrt{7} \\ & + 4\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{31}}{13} + \frac{\sqrt{34}}{7} \\ & + 4\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{37}}{15} \\ & + \left(\sqrt{n}-6\right)\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{10}}{8}\Bigg\}. \end{aligned} \tag{4.1.46}$$

Expanding bracketed terms,

$$\begin{aligned}\mathcal{B}_{MLP-last,lin} &= \sqrt{u}\Bigg\{\frac{\sqrt{14}}{3}+\frac{3}{2}\sqrt{2}+\frac{4}{9}\sqrt{21} \\ &\quad+\left[\frac{2}{5}\sqrt{6}+\frac{2}{5}\sqrt{6}\left(\sqrt{n}-6\right)\right]+\frac{2}{3}\sqrt{7} \\ &\quad+\frac{2}{13}\sqrt{31}\left(\sqrt{n}-6\right)+\frac{\sqrt{34}}{7} \\ &\quad+\frac{2}{15}\sqrt{37}\left(\sqrt{n}-6\right) \\ &\quad+\frac{\sqrt{10}}{8}\left(\frac{n}{2}-6\sqrt{n}+18\right)\Bigg\} \qquad (4.1.47) \\ &= \sqrt{u}\Bigg\{\frac{\sqrt{14}}{3}+\frac{3}{2}\sqrt{2}+\frac{4}{9}\sqrt{21} \\ &\quad+\left[\frac{2}{5}\sqrt{6}+\frac{2}{5}\sqrt{6}\sqrt{n}-\frac{12}{5}\sqrt{6}\right]+\frac{2}{3}\sqrt{7} \\ &\quad+\frac{2}{13}\sqrt{31}\sqrt{n}-\frac{12}{13}\sqrt{31}+\frac{\sqrt{34}}{7} \\ &\quad+\frac{2}{15}\sqrt{37}\sqrt{n}-\frac{12}{15}\sqrt{37} \\ &\quad+\frac{\sqrt{10}}{16}n-\frac{3}{4}\sqrt{10}\sqrt{n}+\frac{9}{4}\sqrt{10}\Bigg\}. \qquad (4.1.48)\end{aligned}$$

Collecting the n-terms together,

$$\begin{aligned}\mathcal{B}_{MLP-last,lin} &= \sqrt{u}\Bigg\{\left[\frac{\sqrt{10}}{16}\right]n+\left[\frac{2}{5}\sqrt{6}+\frac{2}{13}\sqrt{31}+\frac{2}{15}\sqrt{37}-\frac{3}{4}\sqrt{10}\right]n^{\frac{1}{2}} \\ &\quad+\left[\frac{\sqrt{14}}{3}+\frac{3}{2}\sqrt{2}+\frac{4}{9}\sqrt{21}+\frac{2}{5}\sqrt{6}-\frac{12}{5}\sqrt{6}+\frac{2}{3}\sqrt{7}\right. \\ &\quad\left.-\frac{12}{13}\sqrt{31}+\frac{\sqrt{34}}{7}-\frac{4}{5}\sqrt{37}+\frac{9}{4}\sqrt{10}\right]\Bigg\}. \qquad (4.1.49)\end{aligned}$$

Replacing the constant expressions in square brackets with a_9, a_{10} and a_{11} respectively,

$$\mathcal{B}_{MLP-last,lin} = \sqrt{u}\Big[(a_9)\, n + (a_{10})\, n^{\frac{1}{2}} + (a_{11})\Big]. \tag{4.1.50}$$

where constants a_9, a_{10} and a_{11} are:

$$a_9 = \frac{\sqrt{10}}{16} \qquad \approx 0.1976 \tag{4.1.51}$$

$$a_{10} = \frac{2}{5}\sqrt{6} + \frac{2}{13}\sqrt{31} + \frac{2}{15}\sqrt{37} - \frac{3}{4}\sqrt{10} \qquad \approx 0.2757 \tag{4.1.52}$$

$$a_{11} = \frac{\sqrt{14}}{3} + \frac{3}{2}\sqrt{2} + \frac{4}{9}\sqrt{21} + \frac{2}{5}\sqrt{6} - \frac{12}{5}\sqrt{6} + \frac{2}{3}\sqrt{7} - \frac{12}{13}\sqrt{31} + \frac{\sqrt{34}}{7} - \frac{4}{5}\sqrt{37} + \frac{9}{4}\sqrt{10} \qquad \approx 0.2125. \tag{4.1.53}$$

The Badness of MLP Analyzed using Linear Model for $f(d)$

Substituting Equations 4.1.34 and 4.1.50 into 4.1.7,

$$\begin{aligned}\mathcal{B}_{MLP,lin} = {} & sz' \\ & + \sqrt{u}\Big[\quad \Big(2^{-\frac{1}{2}} a_6 + \frac{\sqrt{2}}{8}\Big)\, n + (a_4 + a_7)\, n^{\frac{1}{2}} \\ & \qquad + \Big(a_8 - \sqrt{s} 2^{\frac{v}{2}} a_7 - s 2^v a_6\Big)\Big(2^{\frac{1-v}{4}} s^{-\frac{1}{4}}\Big)\, n^{\frac{1}{4}} \\ & \qquad + \Big(2^{\frac{1}{2}} a_5 - 2^{\frac{1}{2}} a_8\Big)\Big] \\ & + \sqrt{u}\Big[(a_9)\, n + (a_{10})\, n^{\frac{1}{2}} + (a_{11})\Big].\end{aligned} \tag{4.1.54}$$

Collecting the n-terms together,

$$\begin{aligned}\mathcal{B}_{MLP,lin} = sz' + \sqrt{u}\Bigg[&\left(2^{-\frac{1}{2}}a_6 + a_9 + \frac{\sqrt{2}}{8}\right) n \\ &+ \left(a_4 + a_7 + a_{10}\right) n^{\frac{1}{2}} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^v a_6\right)\left(2^{\frac{1-v}{4}} s^{-\frac{1}{4}}\right) n^{\frac{1}{4}} \\ &+ \left(a_{11} + 2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right)\Bigg]. \end{aligned} \tag{4.1.55}$$

But in section 2.3.1, we mentioned that effectively $v = 1$, so,

$$\begin{aligned}\mathcal{B}_{MLP,lin} = sz' + \sqrt{u}\Bigg[&\left(2^{-\frac{1}{2}}a_6 + a_9 + \frac{\sqrt{2}}{8}\right) n \\ &+ \left(a_4 + a_7 + a_{10}\right) n^{\frac{1}{2}} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{1}{2}}a_7 - 2sa_6\right)\left(s^{-\frac{1}{4}}\right) n^{\frac{1}{4}} \\ &+ \left(a_{11} + 2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right)\Bigg]. \end{aligned} \tag{4.1.56}$$

Substituting $u = \frac{w}{\sqrt{n}}$ (2.3.20),

$$\begin{aligned}\mathcal{B}_{MLP,lin} = sz' + \sqrt{\frac{w}{\sqrt{n}}}\Bigg[&\left(2^{-\frac{1}{2}}a_6 + a_9 + \frac{\sqrt{2}}{8}\right) n \\ &+ \left(a_4 + a_7 + a_{10}\right) n^{\frac{1}{2}} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{1}{2}}a_7 - 2sa_6\right)\left(s^{-\frac{1}{4}}\right) n^{\frac{1}{4}} \\ &+ \left(a_{11} + 2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right)\Bigg] \qquad (4.1.57)\end{aligned}$$

$$\begin{aligned}= sz' + \sqrt{w}\Bigg[&\left(2^{-\frac{1}{2}}a_6 + a_9 + \frac{\sqrt{2}}{8}\right) n^{\frac{3}{4}} \\ &+ \left(a_4 + a_7 + a_{10}\right) n^{\frac{1}{4}} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{1}{2}}a_7 - 2sa_6\right)\left(s^{-\frac{1}{4}}\right) \\ &+ \left(a_{11} + 2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right) n^{-\frac{1}{4}}\Bigg]. \qquad (4.1.58)\end{aligned}$$

4.1.3 Analysis Using Sublinear Model for $f(d)$

The Analysis of the Intermediate Levels using Sublinear Model for $f(d)$

From Equation 4.1.16, we substitute $f(d) = u\sqrt{d}$ (2.3.28) and $d_k = 2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}$ (4.1.6),

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} = &\sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{u\sqrt{2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}}} \left[\frac{1}{4}2^k s + \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{s}2^{k/2} + \left(\frac{3}{2} - \sqrt{2}\right)\right] \\ &+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{u\sqrt{2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}}}\left[\frac{1}{4}2^k s + \left(\frac{2}{\sqrt{3}} - 1\right)\sqrt{s}2^{k/2} \right. \\ &\left. + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right)\right]. \end{aligned} \tag{4.1.59}$$

Replacing the constant expressions in round brackets with b_1, $b_2 \ldots b_4$ respectively,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} = &\sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{u\sqrt{2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}}} \left[\frac{1}{4}2^k s + b_1\sqrt{s}2^{k/2} + b_2\right] \\ &+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{u\sqrt{2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}}}\left[\frac{1}{4}2^k s + b_3\sqrt{s}2^{k/2} + b_4\right]. \end{aligned} \tag{4.1.60}$$

If k is even, $\lfloor k/2 \rfloor = k/2$, else $\lfloor k/2 \rfloor = (k-1)/2$,

$$\mathcal{B}_{MLP-int,sub} = \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{u\sqrt{2^{-(k-1)/2}\sqrt{\frac{n}{s}}}} \left[\frac{1}{4}2^k s + b_1\sqrt{s}2^{k/2} + b_2\right] + \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{u\sqrt{2^{-k/2}\sqrt{\frac{n}{s}}}} \left[\frac{1}{4}2^k s + b_3\sqrt{s}2^{k/2} + +b_4\right]. \tag{4.1.61}$$

Since $last$ is even, $m = last - 1$ is odd, but we want the two summations to be over the same number of stages. Hence taking the term when $k = m$ out of the summation over odd k's,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& \left\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m-1} u^{\frac{1}{2}} \left(2^{-(k-1)/2}\right)^{\frac{1}{4}} \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4}2^k s + b_1\sqrt{s}2^{k/2} + b_2\right] \right. \\ & \left. + u^{\frac{1}{2}} \left(2^{-(m-1)/2}\right)^{\frac{1}{4}} \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4}2^m s + b_1\sqrt{s}2^{m/2} + b_2\right] \right\} \\ & + \sum_{\substack{k=v \\ k \text{ even}}}^{m-1} u^{\frac{1}{2}} \left(2^{-k/2}\right)^{\frac{1}{4}} \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4}2^k s + b_3\sqrt{s}2^{k/2} + b_4\right] \end{aligned} \tag{4.1.62}$$

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& \sum_{\substack{k=v \\ k \text{ odd}}}^{m-1} u^{\frac{1}{2}} \left(2^{(1-k)/8}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4}2^k s + b_1\sqrt{s}2^{k/2} + b_2\right] \\ & + u^{\frac{1}{2}} \left(2^{\frac{1-m}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4}2^m s + b_1\sqrt{s}2^{\frac{m}{2}} + b_2\right] \\ & + \sum_{\substack{k=v \\ k \text{ even}}}^{m-1} u^{\frac{1}{2}} \left(2^{-k/8}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4}2^k s + b_3\sqrt{s}2^{k/2} + b_4\right]. \end{aligned} \tag{4.1.63}$$

Change indexing variable $k \to j$, where odd $k = v + 2j - 2$ and even $k = v + 2j - 1$, assuming v is odd,

$$\begin{aligned}
\mathcal{B}_{MLP-int,sub} = & \sum_{j=1}^{\frac{m-v}{2}} u^{\frac{1}{2}} \left(2^{[1-(v+2j-2)]/8}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{v+2j-2} s + b_1 \sqrt{s} 2^{(v+2j-2)/2} + b_2\right] \\
& + \sum_{j=1}^{\frac{m-v}{2}} u^{\frac{1}{2}} \left(2^{-(v+2j-1)/8}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{v+2j-1} s + b_3 \sqrt{s} 2^{(v+2j-1)/2} + b_4\right] \\
& + u^{\frac{1}{2}} \left(2^{\frac{1-m}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{m} s + b_1 \sqrt{s} 2^{\frac{m}{2}} + b_2\right] \qquad (4.1.64) \\
= & \sum_{j=1}^{\frac{m-v}{2}} u^{\frac{1}{2}} \left(2^{\frac{1-v+2-2j}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{v+2j-2} s + b_1 \sqrt{s} 2^{\frac{v+2j-2}{2}} + b_2\right] \\
& + \sum_{j=1}^{\frac{m-v}{2}} u^{\frac{1}{2}} \left(2^{\frac{1-v-2j}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{v+2j-1} s + b_3 \sqrt{s} 2^{\frac{v+2j-1}{2}} + b_4\right] \\
& + u^{\frac{1}{2}} \left(2^{\frac{1-m}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{m} s + b_1 \sqrt{s} 2^{\frac{m}{2}} + b_2\right] \qquad (4.1.65) \\
= & u^{\frac{1}{2}} \left(2^{\frac{1-v}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \sum_{j=1}^{\frac{m-v}{2}} \left\{ 2^{\frac{1-j}{4}} \left[\frac{1}{4} 2^{v+2j-2} s + b_1 \sqrt{s} 2^{\frac{v+2j-2}{2}} + b_2\right] \right. \\
& \qquad \left. + 2^{\frac{-j}{4}} \left[\frac{1}{4} 2^{v+2j-1} s + b_3 \sqrt{s} 2^{\frac{v+2j-1}{2}} + b_4\right] \right\} \\
& + u^{\frac{1}{2}} \left(2^{\frac{1-m}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{m} s + b_1 \sqrt{s} 2^{\frac{m}{2}} + b_2\right] \qquad (4.1.66) \\
= & u^{\frac{1}{2}} \left(2^{\frac{1-v}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \sum_{j=1}^{\frac{m-v}{2}} \left\{ \left[\frac{1}{4} 2^{v+\frac{7}{4}j-\frac{7}{4}} s + b_1 \sqrt{s} 2^{\frac{v}{2}+\frac{3}{4}j-\frac{3}{4}} + b_2 2^{\frac{1-j}{4}}\right] \right. \\
& \qquad \left. + \left[\frac{1}{4} 2^{v+\frac{7}{8}j-1} s + b_3 \sqrt{s} 2^{\frac{v}{2}+\frac{3}{4}j-\frac{1}{2}} + b_4 2^{\frac{-j}{4}}\right] \right\} \\
& + u^{\frac{1}{2}} \left(2^{\frac{1-m}{8}}\right) \left(\frac{n}{s}\right)^{\frac{1}{8}} \left[\frac{1}{4} 2^{m} s + b_1 \sqrt{s} 2^{\frac{m}{2}} + b_2\right]. \qquad (4.1.67)
\end{aligned}$$

Collecting j-terms,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\left(2^{\frac{1-v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\sum_{j=1}^{\frac{m-v}{2}}\Bigg[\left(\frac{1}{4}2^{-\frac{7}{4}}+\frac{1}{4}2^{-1}\right)s2^{v+\frac{7}{4}j} \\ &+\left(b_1 2^{-\frac{3}{4}}+b_3 2^{-\frac{1}{2}}\right)\sqrt{s}2^{\frac{v}{2}+\frac{3}{4}j} \\ &+\left(2b_2+b_4\right)2^{\frac{-j}{4}}\Bigg] \\ &+u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s+b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right]. \end{aligned} \tag{4.1.68}$$

Replacing the constant expressions in round brackets with b_5, b_6 and b_7 respectively,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\left(2^{\frac{1-v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\sum_{j=1}^{\frac{m-v}{2}}\left[b_5 s2^{v+\frac{7}{4}j}+b_6\sqrt{s}2^{\frac{v}{2}+\frac{3}{4}j}+b_7 2^{\frac{-j}{4}}\right] \\ &+u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s+b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right] \end{aligned} \tag{4.1.69}$$

$$\begin{aligned}=& u^{\frac{1}{2}}\left(2^{\frac{1-v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[b_5 s2^v\sum_{j=1}^{\frac{m-v}{2}}2^{\frac{7}{4}j}+b_6\sqrt{s}2^{\frac{v}{2}}\sum_{j=1}^{\frac{m-v}{2}}2^{\frac{3}{4}j}+b_7\sum_{j=1}^{\frac{m-v}{2}}2^{\frac{-j}{4}}\right] \\ &+u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s+b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right]. \end{aligned} \tag{4.1.70}$$

Summing the geometric series,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\left(2^{\frac{1-v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[b_5 s 2^v 2^{\frac{7}{4}}\left(\frac{2^{\frac{7}{4}(\frac{m-v}{2})}-1}{2^{\frac{7}{4}}-1}\right)\right.\\ &+ b_6\sqrt{s}2^{\frac{v}{2}}2^{\frac{3}{4}}\left(\frac{2^{\frac{3}{4}(\frac{m-v}{2})}-1}{2^{\frac{3}{4}}-1}\right)\\ &\left.+ b_7 2^{-\frac{1}{4}}\left(\frac{1-2^{-\frac{1}{4}(\frac{m-v}{2})}}{1-2^{-\frac{1}{4}}}\right)\right]\\ &+ u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s + b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right] \qquad (4.1.71)\\ =& u^{\frac{1}{2}}\left(2^{\frac{1-v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\left(\frac{2^{\frac{7}{4}}b_5}{2^{\frac{7}{4}}-1}s2^v\right)\left(2^{\frac{7}{4}(\frac{m-v}{2})}-1\right)\right.\\ &+ \left(\frac{2^{\frac{3}{4}}b_6}{2^{\frac{3}{4}}-1}\right)\sqrt{s}2^{\frac{v}{2}}\left(2^{\frac{3}{4}(\frac{m-v}{2})}-1\right)\\ &\left.+ \left(\frac{2^{-\frac{1}{4}}b_7}{1-2^{-\frac{1}{4}}}\right)\left(1-2^{-\frac{1}{4}(\frac{m-v}{2})}\right)\right]\\ &+ u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s + b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right]. \qquad (4.1.72)\end{aligned}$$

Replacing the constant expressions in round brackets with b_8, b_9 and b_{10} respectively,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\left(2^{\frac{1-v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\Bigg[b_8 s 2^v\left(2^{\frac{7}{4}(\frac{m-v}{2})}-1\right)\\ &+ b_9\sqrt{s}2^{\frac{v}{2}}\left(2^{\frac{3}{4}(\frac{m-v}{2})}-1\right)\\ &+ b_{10}\left(1-2^{-\frac{1}{4}(\frac{m-v}{2})}\right)\Bigg]\\ &+ u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s + b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right] \qquad (4.1.73)\\ =& u^{\frac{1}{2}}\left(2^{\frac{1-v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\Bigg[b_8 s 2^{\frac{7m+v}{8}} - b_8 s 2^v\\ &+ b_9\sqrt{s}2^{\frac{3m+v}{8}} - b_9\sqrt{s}2^{\frac{v}{2}}\\ &+ b_{10} - b_{10}2^{\frac{v-m}{8}}\Bigg]\\ &+ u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s + b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right]. \qquad (4.1.74)\end{aligned}$$

Distributing $2^{\frac{1-v}{8}}$,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{8}}\Bigg[b_8 s 2^{\frac{1+7m}{8}} - b_8 s 2^{\frac{1+7v}{8}}\\ &+ b_9\sqrt{s}2^{\frac{1+3m}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\\ &+ b_{10}2^{\frac{1-v}{8}} - b_{10}2^{\frac{1-m}{8}}\Bigg]\\ &+ u^{\frac{1}{2}}\left(2^{\frac{1-m}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^m s + b_1\sqrt{s}2^{\frac{m}{2}}+b_2\right]. \qquad (4.1.75)\end{aligned}$$

Substituting $m = \log_2\left(\frac{n}{s}\right) - 1$,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{8}}\Bigg[b_8 s 2^{\frac{1+7\left[\log_2\left(\frac{n}{s}\right)-1\right]}{8}} - b_8 s 2^{\frac{1+7v}{8}} \\ &+ b_9\sqrt{s}2^{\frac{1+3\left[\log_2\left(\frac{n}{s}\right)-1\right]}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}} \\ &+ b_{10}2^{\frac{1-v}{8}} - b_{10}2^{\frac{1-\left[\log_2\left(\frac{n}{s}\right)-1\right]}{8}}\Bigg] \\ &+ u^{\frac{1}{2}}\left(2^{\frac{1-\left[\log_2\left(\frac{n}{s}\right)-1\right]}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{4}2^{\left[\log_2\left(\frac{n}{s}\right)-1\right]}s + b_1\sqrt{s}2^{\frac{\left[\log_2\left(\frac{n}{s}\right)-1\right]}{2}} + b_2\right] \end{aligned} \tag{4.1.76}$$

$$\begin{aligned}=& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{8}}\Bigg[b_8 s 2^{\frac{7}{8}\log_2\left(\frac{n}{s}\right)-\frac{3}{4}} - b_8 s 2^{\frac{1+7v}{8}} \\ &+ b_9\sqrt{s}2^{\frac{3}{8}\log_2\left(\frac{n}{s}\right)-\frac{1}{4}} - b_9\sqrt{s}2^{\frac{1+3v}{8}} \\ &+ b_{10}2^{\frac{1-v}{8}} - b_{10}2^{\frac{1}{4}-\frac{1}{8}\log_2\left(\frac{n}{s}\right)}\Bigg] \\ &+ u^{\frac{1}{2}}\left(2^{\frac{1}{4}-\frac{1}{8}\log_2\left(\frac{n}{s}\right)}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{8}\left(\frac{n}{s}\right)s + b_1\sqrt{s}2^{-\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{2}} + b_2\right] \end{aligned} \tag{4.1.77}$$

$$\begin{aligned}=& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{8}}\Bigg[b_8 s 2^{-\frac{3}{4}}\left(\frac{n}{s}\right)^{\frac{7}{8}} - b_8 s 2^{\frac{1+7v}{8}} \\ &+ b_9\sqrt{s}2^{-\frac{1}{4}}\left(\frac{n}{s}\right)^{\frac{3}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}} \\ &+ b_{10}2^{\frac{1-v}{8}} - b_{10}2^{\frac{1}{4}}\left(\frac{n}{s}\right)^{-\frac{1}{8}}\Bigg] \\ &+ u^{\frac{1}{2}}\left[2^{\frac{1}{4}}\left(\frac{n}{s}\right)^{-\frac{1}{8}}\right]\left(\frac{n}{s}\right)^{\frac{1}{8}}\left[\frac{1}{8}\left(\frac{n}{s}\right)s + b_1\sqrt{s}2^{-\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{2}} + b_2\right]. \end{aligned} \tag{4.1.78}$$

Grouping non-n-terms at the end of each subexpression,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\Bigg[b_8 s 2^{-\frac{3}{4}}\left(\frac{n}{s}\right) + b_9\sqrt{s}2^{-\frac{1}{4}}\left(\frac{n}{s}\right)^{\frac{1}{2}} + \\ & \left(b_{10}2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}} - b_{10}2^{\frac{1}{4}}\Bigg] \\ & + u^{\frac{1}{2}}2^{\frac{1}{4}}\left[\frac{1}{8}n + b_1 2^{-\frac{1}{2}}n^{\frac{1}{2}} + b_2\right]. \end{aligned} \tag{4.1.79}$$

Consolidating n-terms,

$$\begin{aligned}\mathcal{B}_{MLP-int,sub} =& u^{\frac{1}{2}}\Bigg[\left(b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}}\right)n + \left(b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}}\right)n^{\frac{1}{2}} + \\ & \left(b_{10}2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}} + \left(b_2 - b_{10}\right)2^{\frac{1}{4}}\Bigg] \end{aligned} \tag{4.1.80}$$

where constants $b_1, b_2, \ldots b_{11}$ are:

$$b_1 = 1 - \frac{1}{\sqrt{2}} \tag{4.1.81}$$

$$b_2 = \frac{3}{2} - \sqrt{2} \tag{4.1.82}$$

$$b_3 = \frac{2}{\sqrt{3}} - 1 \tag{4.1.83}$$

$$b_4 = \sqrt{2} - \frac{4}{\sqrt{3}} + 1 \tag{4.1.84}$$

$$b_5 = \frac{1}{4}2^{-\frac{7}{4}} + \frac{1}{4}2^{-1} \qquad \approx 0.1993 \tag{4.1.85}$$

$$b_6 = b_1 2^{-\frac{3}{4}} + b_3 2^{-\frac{1}{2}} \tag{4.1.86}$$

$$b_7 = 2b_2 + b_4 \tag{4.1.87}$$

$$b_8 = \frac{2^{\frac{7}{4}}b_5}{2^{\frac{7}{4}} - 1} \qquad \approx 0.2837 \tag{4.1.88}$$

$$b_9 = \frac{2^{\frac{3}{4}} b_6}{2^{\frac{3}{4}} - 1} \tag{4.1.89}$$

$$b_{10} = \frac{2^{-\frac{1}{4}} b_7}{1 - 2^{-\frac{1}{4}}}. \tag{4.1.90}$$

The Analysis of the Last Level using Sublinear Model for $f(d)$

$$\mathcal{B}_{MLP-last,sub} = \sum_{c=1}^{16} n_{last,c} \frac{\sqrt{\sum_d c_d f(d)}}{c}. \tag{4.1.91}$$

Substituting $f(d) = u\sqrt{d}$ (2.3.28),

$$\mathcal{B}_{MLP-last,sub} = \sqrt{u} \sum_{c=1}^{16} n_{last,c} \frac{\sqrt{\sum_d c_d \sqrt{d}}}{c}. \tag{4.1.92}$$

Substituting the expressions for $n_{k,c}$ from Table 4.2 ,

$$\begin{aligned}
\mathcal{B}_{MLP-last,sub} = \sqrt{u} \Bigg\{ & 2 \times \frac{\sqrt{2+4\sqrt{3}}}{6} + 4 \times \frac{\sqrt{3+5\sqrt{3}}}{8} + 4 \times \frac{\sqrt{3+6\sqrt{3}}}{9} \\
& + \left[2 + 4 \left(\frac{\sqrt{n}-6}{2} \right) \right] \frac{\sqrt{3+7\sqrt{3}}}{10} + 4 \times \frac{\sqrt{4+8\sqrt{3}}}{12} \\
& + 4 \left(\frac{\sqrt{n}-6}{2} \right) \frac{\sqrt{4+9\sqrt{3}}}{13} + 2 \times \frac{\sqrt{4+10\sqrt{3}}}{14} \\
& + 4 \left(\frac{\sqrt{n}-6}{2} \right) \frac{\sqrt{4+11\sqrt{3}}}{15} \\
& + \left(\sqrt{n} - 6 \right) \left(\frac{\sqrt{n}-6}{2} \right) \frac{\sqrt{4+12\sqrt{3}}}{16} \Bigg\}
\end{aligned} \tag{4.1.93}$$

$$\begin{aligned}\mathcal{B}_{MLP-last,sub} = \sqrt{u}\Bigg\{ & \frac{1}{3}\sqrt{2+4\sqrt{3}} + \frac{1}{4}\sqrt{3+5\sqrt{3}} + \frac{4}{9} \times \sqrt{3+6\sqrt{3}} \\ & + \frac{\sqrt{n}}{5}\sqrt{3+7\sqrt{3}} - \sqrt{3+7\sqrt{3}} + \frac{1}{3}\sqrt{4+8\sqrt{3}} \\ & + \frac{2\sqrt{n}}{13}\sqrt{4+9\sqrt{3}} - \frac{6}{13}\sqrt{4+9\sqrt{3}} + \frac{1}{7}\sqrt{4+10\sqrt{3}} \\ & + \frac{2\sqrt{n}}{15}\sqrt{4+11\sqrt{3}} - \frac{6}{15}\sqrt{4+11\sqrt{3}} \\ & + \frac{n}{32}\sqrt{4+12\sqrt{3}} - \frac{3\sqrt{n}}{8}\sqrt{4+12\sqrt{3}} - \frac{9}{8}\sqrt{4+12\sqrt{3}}\Bigg\}. \end{aligned} \tag{4.1.94}$$

Grouping n-terms together,

$$\begin{aligned}\mathcal{B}_{MLP-last,sub} = \sqrt{u}\Bigg\{ & \left(\frac{1}{32}\sqrt{4+12\sqrt{3}}\right) n \\ & + \left(\frac{1}{5}\sqrt{3+7\sqrt{3}} + \frac{2}{13}\sqrt{4+9\sqrt{3}} \right. \\ & \quad \left. - \frac{3}{8}\sqrt{4+12\sqrt{3}} + \frac{2}{15}\sqrt{4+11\sqrt{3}}\right)\sqrt{n} \\ & + \left(\frac{1}{3}\sqrt{2+4\sqrt{3}} + \frac{1}{4}\sqrt{3+5\sqrt{3}} + \frac{4}{9}\sqrt{3+6\sqrt{3}} \right. \\ & \quad - \sqrt{3+7\sqrt{3}} + \frac{1}{3}\sqrt{4+8\sqrt{3}} - \frac{6}{13}\sqrt{4+9\sqrt{3}} \\ & \quad \left. + \frac{1}{7}\sqrt{4+10\sqrt{3}} - \frac{6}{15}\sqrt{4+11\sqrt{3}} - \frac{9}{8}\sqrt{4+12\sqrt{3}}\right)\Bigg\}. \end{aligned} \tag{4.1.95}$$

Replacing constant expressions in round brackets with b_{11}, b_{12} and b_{13},

$$\mathcal{B}_{MLP-last,sub} = \sqrt{u}\left\{ b_{11}n + b_{12}\sqrt{n} + b_{13} \right\} \tag{4.1.96}$$

where constants b_{11}, b_{12} and b_{13} are:

$$b_{11} = \frac{1}{32}\sqrt{4+12\sqrt{3}} \qquad \approx 0.1556 \tag{4.1.97}$$

$$\begin{aligned} b_{12} = {} & \frac{1}{5}\sqrt{3+7\sqrt{3}} + \frac{2}{13}\sqrt{4+9\sqrt{3}} \\ & - \frac{3}{8}\sqrt{4+12\sqrt{3}} + \frac{2}{15}\sqrt{4+11\sqrt{3}} \end{aligned} \tag{4.1.98}$$

$$\begin{aligned} b_{13} = {} & \frac{1}{3}\sqrt{2+4\sqrt{3}} + \frac{1}{4}\sqrt{3+5\sqrt{3}} + \frac{4}{9}\sqrt{3+6\sqrt{3}} - \sqrt{3+7\sqrt{3}} \\ & + \frac{1}{3}\sqrt{4+8\sqrt{3}} - \frac{6}{13}\sqrt{4+9\sqrt{3}} + \frac{1}{7}\sqrt{4+10\sqrt{3}} \\ & - \frac{6}{15}\sqrt{4+11\sqrt{3}} - \frac{9}{8}\sqrt{4+12\sqrt{3}}. \end{aligned} \tag{4.1.99}$$

The Badness of MLP Analyzed using Sublinear Model for $f(d)$

$$\mathcal{B}_{MLP,sub} = sz' + \mathcal{B}_{MLP-int,sub} + \mathcal{B}_{MLP-last,sub}. \tag{4.1.100}$$

Substituting Equations 4.1.80 and 4.1.96,

$$\begin{aligned}\mathcal{B}_{MLP,sub} =& sz' \\ &+ u^{\frac{1}{2}}\left[b_{11}n + b_{12}\sqrt{n} + b_{13}\right] \\ &+ u^{\frac{1}{2}}\left[\left(b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}}\right)n + \left(b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}}\right)n^{\frac{1}{2}} + \right. \\ &\quad \left.\left(b_{10}2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}} + \left(b_2 - b_{10}\right)2^{\frac{1}{4}}\right].\end{aligned} \tag{4.1.101}$$

Grouping n-terms together,

$$\begin{aligned}\mathcal{B}_{MLP,sub} =& sz' \\ &+ u^{\frac{1}{2}}\left[\left(b_{11} + b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}}\right)n + \left(b_{12} + b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}}\right)n^{\frac{1}{2}}\right. \\ &\quad + \left(b_{10}2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}} \\ &\quad \left. + \left(b_2 2^{\frac{1}{4}} - b_{10}2^{\frac{1}{4}} + b_{13}\right)\right].\end{aligned} \tag{4.1.102}$$

Substituting $u = \frac{w}{\sqrt[4]{n}}$ (2.3.20)

$$\begin{aligned}\mathcal{B}_{MLP,sub} =& sz' \\ &+ \left(\frac{w}{\sqrt[4]{n}}\right)^{\frac{1}{2}} \left[\left(b_{11} + b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}}\right) n + \left(b_{12} + b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}}\right) n^{\frac{1}{2}}\right. \\ &\qquad + \left(b_{10} 2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}} \\ &\qquad \left. + \left(b_2 2^{\frac{1}{4}} - b_{10}2^{\frac{1}{4}} + b_{13}\right)\right] \qquad (4.1.103) \\ =& sz' \\ &+ w^{\frac{1}{2}}\left[\left(b_{11} + b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}}\right) n^{\frac{7}{8}} + \left(b_{12} + b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}}\right) n^{\frac{3}{8}}\right. \\ &\qquad + \left(b_{10} 2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right) s^{-\frac{1}{8}} \\ &\qquad \left. + \left(b_2 2^{\frac{1}{4}} - b_{10}2^{\frac{1}{4}} + b_{13}\right) n^{-\frac{1}{8}}\right]. \qquad (4.1.104)\end{aligned}$$

4.1.4 The Badness of MLP

Summarizing the last two sections, we tabulate the dominant term in our two expressions for the badness of MLP together with those for raster scan restricted to distance 2 in Table 4.3. MLP seems to be worse than raster scan

Pixel Ordering	Linear Model	Sublinear Model
MLP	$0.608\sqrt{w}n^{\frac{3}{4}}$	$0.473\sqrt{w}n^{\frac{7}{8}}$
Raster Scan $l = 2$	$0.527\sqrt{w}n^{\frac{3}{4}}$	$0.464\sqrt{w}n^{\frac{7}{8}}$

Table 4.3: Comparison of MLP and raster scan ($l = 2$).

even at $l = 2$ and with $f(d) = u\sqrt{u}$ (recall that raster scan's badness decreases with increasing l), but note that this comparison is not entirely fair. Raster

scan at $l = 2$ uses 4 neighbors at at-most distance 2 for almost every prediction, whereas MLP uses 16 neighbors of distance at-most 3 for about half of the total number of predictions and 4 neighbors of at least distance 2 for the other half. In order to make a fairer comparison, we propose a generalized MLP (GMLP) method that removes the restriction that the intermediate levels must use a 4-neighbor pixel choice and that the final level must use a 16-neighbor pixel choice. A particular case of this GMLP ordering where only 6-neighbor pixel choices are allowed is analyzed next.

4.2 GMLP6 : GMLP Pixel Ordering Restricted to 6 Neighbors

4.2.1 Overview

In this section we examine a special case of the generalized MLP method restricted to using exactly 6 neighbors at all levels. The reason we impose this restriction is to enable a fairer comparison of MLP and raster scan pixel ordering. Comparing the original MLP method which uses a 4-neighbor pixel choice at the intermediate levels and a 16-neighbor pixel choice at the last level to a raster scan ordering that consistently uses a fixed number of neighbors is odd. Restricting the number of neighbors in the pixel choice each ordering could use will hopefully enable the pixel ordering inherent in the method to be better evaluated. The choice of a 6-neighbor pixel choice (as opposed to 4-neighbor or k-neighbor) is arbitrary.

The pixel choice of GMLP6 is shown in Figure 4.3. A point of departure

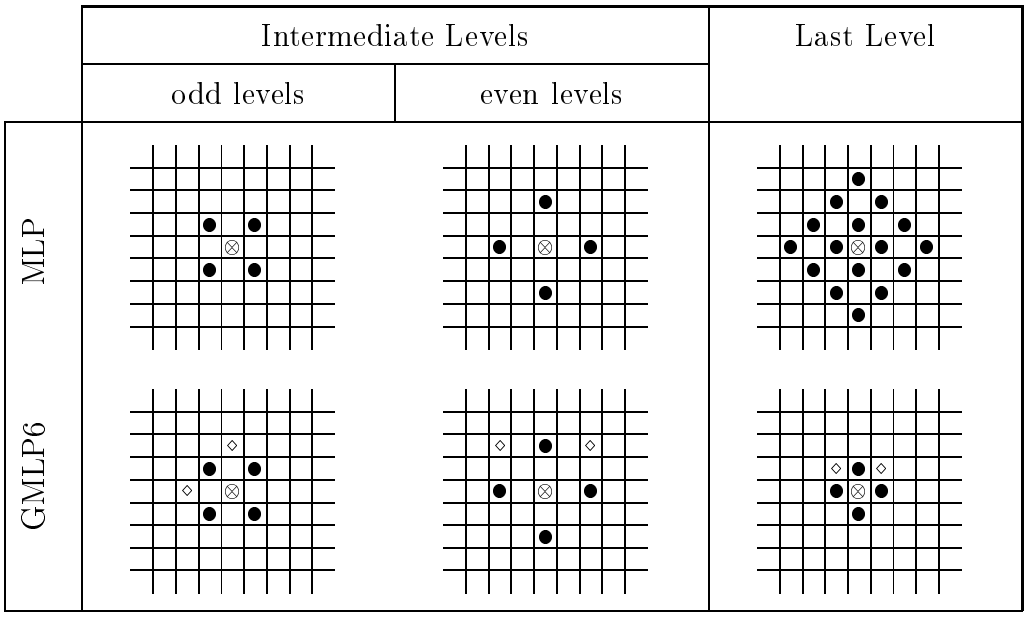


Figure 4.3: The pixel choice used in MLP and GMLP limited to 6 neighbors. The crossed circle ⊗ represents the pixel to be predicted. The hollow circles ∘ denote the neighbors predicted at the previous levels. The diamonds ⋄ denote the neighbors that are predicted at the current level in the usual top-down, left-right sequence.

from MLP is that in addition to neighbors predicted at previous levels, this modified scheme also uses neighbors that are predicted prior to the pixel at the current level. Moreover, the even levels uses neighbors that are at distance $2d_k$, where d_k is the distance of the neighbors used at level k in the baseline MLP algorithm.

An implication of using neighbors at distance $2d_k$ is that the number of starting pixels s would have to be quadrupled compared to that of normal MLP in order to make the first level distances d_1 within the threshold α.

The expressions for the number of pixels processed at level k using c neighbors, $n_{k,c}$, are derived in a similar manner as the previous section only that in this particular pixel ordering, there are cases when the same total number of neighbors c may have differing constituent c_d's (the number of neighbors at distance d). The values are present in Table 4.4 and Table 4.5. The restriction in neighborhood size has also made the last level indistinguishable from the intermediate levels; hence the analysis can be performed in one step.

$c = c_d$	$n_{k,c}$ for odd k
1	1
2	$\sqrt{n_k} - 1$
3	0
4	$\sqrt{n_k} - 1$
5	$\sqrt{n_k} - 1$
6	$(\sqrt{n_k} - 2)(\sqrt{n_k} - 1)$

Table 4.4: $n_{k,c}$ for GMLP6 odd levels.

c	c_d	c_{2d}	$n_{k,c}$ for even k
2	2	0	1
3	3	0	$\sqrt{n_{k-1}} - 1$
3	2	1	1
4	3	1	$2\left(\sqrt{n_{k-1}} - 1\right)$
5	3	2	$\sqrt{n_{k-1}} - 1$
6	4	2	$2\left(\sqrt{n_{k-1}} - 1\right)^2$
otherwise			0

Table 4.5: $n_{k,c}$ for GMLP6 even levels.

4.2.2 The Analysis of GMLP6

Summing from an arbitrary level v to m,

$$\mathcal{B}_{GMLP6} = \sum_{k=v}^{m} \left\{ \sum_{\text{all } c} n_{k,c} \frac{\sqrt{c_{d_k} f(d_k) + c_{2d_k} f(2d_k)}}{c} \right\}. \tag{4.2.105}$$

Substituting $f(d_k) = ud_k$ (2.3.18),

$$\mathcal{B}_{GMLP6} = \sum_{k=v}^{m} \left\{ \sum_{\text{all } c} n_{k,c} \frac{\sqrt{ud_k \left(c_{d_k} + 2c_{2d_k}\right)}}{c} \right\} \tag{4.2.106}$$

$$= \sqrt{u} \sum_{k=v}^{m} \left\{ \sqrt{d_k} \sum_{\text{all } c} n_{k,c} \frac{\sqrt{c_{d_k} + 2c_{2d_k}}}{c} \right\}. \tag{4.2.107}$$

Splitting the summation into odd and even k,

$$\mathcal{B}_{GMLP6} = \sqrt{u} \Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{d_k} \sum_{c=1}^{6} n_{k,c} \frac{\sqrt{c_{d_k} + 2c_{2d_k}}}{c} + \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{d_k} \sum_{c=1}^{6} n_{k,c} \frac{\sqrt{c_{d_k} + 2c_{2d_k}}}{c} \Bigg\}. \tag{4.2.108}$$

The odd levels use neighbors that are at distance d_k only; hence $c_{2d_k} = 0$ and $c_{d_k} = c$ for odd k,

$$\mathcal{B}_{GMLP6} = \sqrt{u}\Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{d_k} \sum_{c=1}^{6} n_{k,c} \frac{1}{\sqrt{c}} + \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{d_k} \sum_{c=1}^{6} n_{k,c} \frac{\sqrt{c_{d_k} + 2c_{2d_k}}}{c} \Bigg\}. \tag{4.2.109}$$

Substituting the expressions for $n_{k,c}$ from Table 4.4 and Table 4.5,

$$\begin{aligned}
\mathcal{B}_{GMLP6} =&\sqrt{u}\Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{d_k}\Bigg[1\frac{1}{\sqrt{1}} + (\sqrt{n_k}-1)\,\frac{1}{\sqrt{2}} + (\sqrt{n_k}-1)\,\frac{1}{\sqrt{4}} \\
&\qquad + (\sqrt{n_k}-1)\,\frac{1}{\sqrt{5}} + (\sqrt{n_k}-2)\,(\sqrt{n_k}-1)\,\frac{1}{\sqrt{6}}\Bigg] \\
&+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{d_k}\Bigg[\frac{\sqrt{2+2\cdot 0}}{2} + (\sqrt{n_{k-1}}-1)\,\frac{\sqrt{3+2\cdot 0}}{3} + \frac{\sqrt{2+2\cdot 1}}{3} \\
&\qquad + 2\,(\sqrt{n_{k-1}}-1)\,\frac{\sqrt{3+2\cdot 1}}{4} + (\sqrt{n_{k-1}}-1)\,\frac{\sqrt{3+2\cdot 2}}{5} \\
&\qquad + 2\,(\sqrt{n_{k-1}}-1)^2\,\frac{\sqrt{4+2\cdot 2}}{6}\Bigg]\Bigg\} \qquad (4.2.110) \\
=&\sqrt{u}\Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{d_k}\Bigg[1 + \frac{1}{\sqrt{2}}\left(n_k^{\frac{1}{2}}\right) - \frac{1}{\sqrt{2}} + \frac{1}{2}\left(n_k^{\frac{1}{2}}\right) - \frac{1}{2} \\
&\qquad + \frac{1}{\sqrt{5}}\left(n_k^{\frac{1}{2}}\right) - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}}\,(n_k) - \frac{3}{\sqrt{6}}\left(n_k^{\frac{1}{2}}\right) + \frac{2}{\sqrt{6}}\Bigg] \\
&+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{d_k}\Bigg[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\left(n_{k-1}^{\frac{1}{2}}\right) - \frac{1}{\sqrt{3}} + \frac{2}{3} \\
&\qquad + \frac{\sqrt{5}}{2}\left(n_{k-1}^{\frac{1}{2}}\right) - \frac{\sqrt{5}}{2} + \frac{\sqrt{7}}{5}\left(n_{k-1}^{\frac{1}{2}}\right) - \frac{\sqrt{7}}{5} \\
&\qquad + \frac{2\sqrt{2}}{3}\,(n_{k-1}) - \frac{4\sqrt{2}}{3}\left(n_{k-1}^{\frac{1}{2}}\right) + \frac{2\sqrt{2}}{3}\Bigg]\Bigg\}. \qquad (4.2.111)
\end{aligned}$$

Collecting the n-terms together,

$$\begin{aligned}\mathcal{B}_{GMLP6} =&\sqrt{u}\Bigg\{\sum_{\substack{k=v\\ k \text{ odd}}}^{m}\sqrt{d_k}\Bigg[\left(\frac{1}{\sqrt{6}}\right)n_k+\left(\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{\sqrt{5}}-\frac{3}{\sqrt{6}}\right)n_k^{\frac{1}{2}}\\ &\qquad+\left(\frac{1}{2}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{6}}\right)\Bigg]\\ &+\sum_{\substack{k=v\\ k \text{ even}}}^{m}\sqrt{d_k}\Bigg[\left(\frac{2\sqrt{2}}{3}\right)n_{k-1}+\left(\frac{1}{\sqrt{3}}+\frac{\sqrt{5}}{2}+\frac{\sqrt{7}}{5}-\frac{4\sqrt{2}}{3}\right)n_{k-1}^{\frac{1}{2}}\\ &\qquad+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{2}{3}-\frac{\sqrt{5}}{2}-\frac{\sqrt{7}}{5}+\frac{2\sqrt{2}}{3}\right)\Bigg]\Bigg\}.\end{aligned} \tag{4.2.112}$$

Replacing the constant expressions in round brackets with $g_1 \ldots g_6$ respectively,

$$\begin{aligned}\mathcal{B}_{GMLP6} =&\sqrt{u}\Bigg\{\sum_{\substack{k=v\\ k \text{ odd}}}^{m}\sqrt{d_k}\Big[g_1 n_k+g_2 n_k^{\frac{1}{2}}+g_3\Big]\\ &+\sum_{\substack{k=v\\ k \text{ even}}}^{m}\sqrt{d_k}\Big[g_4 n_{k-1}+g_5 n_{k-1}^{\frac{1}{2}}+g_6\Big]\Bigg\}.\end{aligned} \tag{4.2.113}$$

Substituting $n_k = 2^{k-1}s$ (see (4.1.2)), $d_k = 2^{-k/2+1/2}\sqrt{\frac{n}{s}}$ for odd k and $d_k = 2^{-k/2}\sqrt{\frac{n}{s}}$ for even k (4.1.6),

$$\begin{aligned}\mathcal{B}_{GMLP6} =&\sqrt{u}\Bigg\{\sum_{\substack{k=v\\ k \text{ odd}}}^{m}\sqrt{2^{-\frac{k}{2}+\frac{1}{2}}\sqrt{\frac{n}{s}}}\Big[\;\; g_1 2^{k-1}s+g_2 2^{\frac{1}{2}(k-1)}s^{\frac{1}{2}}+g_3\Big]\\ &+\sum_{\substack{k=v\\ k \text{ even}}}^{m}\sqrt{2^{-\frac{k}{2}}\sqrt{\frac{n}{s}}}\Big[\;\; g_4 2^{k-2}s+g_5 2^{\frac{1}{2}k-1}s^{\frac{1}{2}}+g_6\Big]\Bigg\}.\end{aligned} \tag{4.2.114}$$

Change indexing variable $k \to j$, where odd $k = v + 2j - 2$ and even $k = v + 2j - 1$, assuming m is even and v is odd,

$$\begin{aligned}\mathcal{B}_{GMLP6} =& u^{\frac{1}{2}} \left(\frac{n}{s}\right)^{\frac{1}{4}} \left\{ \sum_{j=1}^{\frac{m-v+1}{2}} \sqrt{2^{-\frac{v+2j-2}{2}+\frac{1}{2}}} \left[g_1 2^{v+2j-3} s + g_2 2^{\frac{1}{2}(v+2j-3)} s^{\frac{1}{2}} + g_3 \right] \right. \\ & \left. + \sum_{j=1}^{\frac{m-v+1}{2}} \sqrt{2^{-\frac{v+2j-1}{2}}} \left[g_4 2^{v+2j-3} s + g_5 2^{\frac{1}{2}(v+2j-1)-1} s^{\frac{1}{2}} + g_6 \right] \right\} \qquad (4.2.115) \\ =& u^{\frac{1}{2}} \left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}} \left\{ \sum_{j=1}^{\frac{m-v+1}{2}} \left[2^{\frac{1-j}{2}} \left(g_1 2^{v+2j-3} s + g_2 2^{\frac{1}{2}v+j-\frac{3}{2}} s^{\frac{1}{2}} + g_3 \right) \right. \right. \\ & \left. \left. + 2^{\frac{-j}{2}} \left(g_4 2^{v+2j-3} s + g_5 2^{\frac{1}{2}v+j-\frac{3}{2}} s^{\frac{1}{2}} + g_6 \right) \right] \right\}. \qquad (4.2.116)\end{aligned}$$

Collecting j-terms,

$$\begin{aligned}\mathcal{B}_{GMLP6} =& u^{\frac{1}{2}} \left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}} \left\{ \sum_{j=1}^{\frac{m-v+1}{2}} \left[\left(2^{\frac{-5}{2}} g_1 + 2^{-3} g_4 \right) 2^v s 2^{\frac{3}{2}j} \right. \right. \\ & + \left(2^{-1} g_2 + 2^{\frac{-3}{2}} g_5 \right) 2^{\frac{v}{2}} s^{\frac{1}{2}} 2^{\frac{j}{2}} \\ & \left. \left. + \left(2^{\frac{1}{2}} g_3 + g_6 \right) 2^{\frac{-j}{2}} \right] \right\}. \qquad (4.2.117)\end{aligned}$$

Summing the geometric series $2^{\frac{3}{2}j}$, $2^{\frac{j}{2}}$ and $2^{\frac{-j}{2}}$,

$$\begin{aligned}\mathcal{B}_{GMLP6} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{\left(2^{\frac{-5}{2}}g_1+2^{-3}g_4\right)2^v s 2^{\frac{3}{2}}\left(\frac{2^{\frac{3}{2}(\frac{m-v+1}{2})}-1}{2^{\frac{3}{2}}-1}\right)\\ &+\left(2^{-1}g_2+2^{\frac{-3}{2}}g_5\right)2^{\frac{v}{2}}s^{\frac{1}{2}}2^{\frac{1}{2}}\left(\frac{2^{\frac{1}{2}(\frac{m-v+1}{2})}-1}{2^{\frac{1}{2}}-1}\right)\\ &+\left(2^{\frac{1}{2}}g_3+g_6\right)2^{-\frac{1}{2}}\left(\frac{1-2^{-\frac{1}{2}(\frac{m-v+1}{2})}}{1-2^{-\frac{1}{2}}}\right)\Bigg\}\end{aligned} \tag{4.2.118}$$

$$\begin{aligned}=& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{\left[\left(2^{\frac{-5}{2}}g_1+2^{-3}g_4\right)\left(\frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}}-1}\right)\right]2^v s\left(2^{\frac{3}{2}(\frac{m-v+1}{2})}-1\right)\\ &+\left[\left(2^{-1}g_2+2^{\frac{-3}{2}}g_5\right)\left(\frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1}\right)\right]2^{\frac{v}{2}}s^{\frac{1}{2}}\left(2^{\frac{1}{2}(\frac{m-v+1}{2})}-1\right)\\ &+\left[\left(2^{\frac{1}{2}}g_3+g_6\right)\left(\frac{2^{-\frac{1}{2}}}{1-2^{-\frac{1}{2}}}\right)\right]\left(1-2^{-\frac{1}{2}(\frac{m-v+1}{2})}\right)\Bigg\}.\end{aligned} \tag{4.2.119}$$

Replacing constant expressions in $[\ldots]$ again,

$$\begin{aligned}\mathcal{B}_{GMLP6} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{g_7 2^v s\left(2^{\frac{3}{2}(\frac{m-v+1}{2})}-1\right)\\ &+g_8 2^{\frac{v}{2}}s^{\frac{1}{2}}\left(2^{\frac{1}{2}(\frac{m-v}{2})}-1\right)\\ &+g_9\left(1-2^{-\frac{1}{2}(\frac{m-v}{2})}\right)\Bigg\}\end{aligned} \tag{4.2.120}$$

$$\begin{aligned}\mathcal{B}_{GMLP6} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{g_7 2^v s 2^{\frac{3}{4}(m-v+1)} - g_7 2^v s \\ &+ g_8 2^{\frac{v}{2}} s^{\frac{1}{2}} 2^{\frac{1}{4}(m-v+1)} - g_8 2^{\frac{v}{2}} s^{\frac{1}{2}} \\ &+ g_9 - g_9 2^{-\frac{1}{4}(m-v+1)}\Bigg\} \qquad (4.2.121)\\ =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\Bigg\{2g_7 s 2^{\frac{3}{4}m} + 2^{\frac{1}{2}} g_8 s^{\frac{1}{2}} 2^{\frac{1}{4}m} \\ &+ \left(g_9 - g_7 2^v s - g_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} - g_9 2^{-\frac{1}{4}m}\Bigg\}. \qquad (4.2.122)\end{aligned}$$

Substituting $m = \log_2\left(\frac{n}{s}\right)$,

$$\begin{aligned}\mathcal{B}_{GMLP6} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\Bigg\{2g_7 s 2^{\frac{3}{4}\log_2\left(\frac{n}{s}\right)} + 2^{\frac{1}{2}} g_8 s^{\frac{1}{2}} 2^{\frac{1}{4}\log_2\left(\frac{n}{s}\right)} \\ &+ \left(g_9 - g_7 2^v s - g_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} - g_9 2^{-\frac{1}{4}\log_2\left(\frac{n}{s}\right)}\Bigg\} \qquad (4.2.123)\\ =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\Bigg\{2g_7 s\left(\frac{n}{s}\right)^{\frac{3}{4}} + 2^{\frac{1}{2}} g_8 s^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} \\ &+ \left(g_9 - g_7 2^v s - g_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} - g_9\left(\frac{n}{s}\right)^{-\frac{1}{4}}\Bigg\} \qquad (4.2.124)\\ =& u^{\frac{1}{2}}\Bigg\{2g_7 n + 2^{\frac{1}{2}} g_8 n^{\frac{1}{2}} \\ &+ \left(g_9 - g_7 2^v s - g_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} n^{\frac{1}{4}} s^{-\frac{1}{4}} - g_9\Bigg\}. \qquad (4.2.125)\end{aligned}$$

But in Section 2.3.1, we mentioned that effectively $v = 1$, so,

$$\mathcal{B}_{GMLP6} = u^{\frac{1}{2}} \left\{ 2g_7 n + 2^{\frac{1}{2}} g_8 n^{\frac{1}{2}} + \left(g_9 - 2g_7 s - 2^{\frac{1}{2}} g_8 s^{\frac{1}{2}} \right) s^{-\frac{1}{4}} n^{\frac{1}{4}} - g_9 \right\}. \tag{4.2.126}$$

Substituting $u = \frac{w}{\sqrt{n}}$ (2.3.20),

$$\mathcal{B}_{GMLP6} = w^{\frac{1}{2}} \left\{ 2g_7 n^{\frac{3}{4}} + 2^{\frac{1}{2}} g_8 n^{\frac{1}{4}} + \left(g_9 - 2g_7 s - 2^{\frac{1}{2}} g_8 s^{\frac{1}{2}} \right) s^{-\frac{1}{4}} - g_9 n^{-\frac{1}{4}} \right\} \tag{4.2.127}$$

where,

$$g_1 = \frac{1}{\sqrt{6}} \tag{4.2.128}$$

$$g_2 = \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{5}} - \frac{3}{\sqrt{6}} \tag{4.2.129}$$

$$g_3 = \frac{1}{2} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{6}} \tag{4.2.130}$$

$$g_4 = \frac{2\sqrt{2}}{3} \tag{4.2.131}$$

$$g_5 = \frac{1}{\sqrt{3}} + \frac{\sqrt{5}}{2} + \frac{\sqrt{7}}{5} - \frac{4\sqrt{2}}{3} \tag{4.2.132}$$

$$g_6 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{2}{3} - \frac{\sqrt{5}}{2} - \frac{\sqrt{7}}{5} + \frac{2\sqrt{2}}{3} \tag{4.2.133}$$

$$g_7 = \left(2^{\frac{-5}{2}} g_1 + 2^{-3} g_4 \right) \left(\frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}} - 1} \right) \qquad \approx 0.2939 \tag{4.2.134}$$

$$g_8 = \left(2^{-1} g_2 + 2^{\frac{-3}{2}} g_5 \right) \left(\frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}} - 1} \right) \tag{4.2.135}$$

$$g_9 = \left(2^{\frac{1}{2}} g_3 + g_6 \right) \left(\frac{2^{-\frac{1}{2}}}{1 - 2^{-\frac{1}{2}}} \right). \tag{4.2.136}$$

Hence the dominant term in $\mathcal{B}_{GMLP6}$ is $0.588\sqrt{w}n^{\frac{3}{4}}$ which is much worse than the $0.441\sqrt{w}n^{\frac{3}{4}}$ of raster scan with 6 neighbors ($l = 3$). Since GMLP6 is still worse than raster scan, we analyze another case of GMLP that uses a pixel choice of even more neighbors next.

4.3 GMLP20: A GMLP Pixel Ordering Using 20 neighbors

Having analyzed MLP and GMLP6, the next question would be how using more neighbors would affect its badness. In this section we analyze the GMLP20 method which, in addition to MLP pixel ordering, uses a pixel choice that include all available neighbors within a distance of $2d_k$ (d_k is the distance of the neighbors used at level k) for the intermediate levels and within a distance of 3 for the last level. The positions of the neighbors are shown in Figure 4.4. As with GMLP6, GMLP20 also uses neighbors that are predicted prior to the pixel at the current level.

As with GMLP6, the number of starting pixels s must be quadrupled that of standard MLP in order to make the first level distances d_1 within the threshold α.

The expressions for the number of pixels processed at level k using c neighbors, $n_{k,c}$, are present in Table 4.6, Table 4.7 and Table 4.8. Again, there are cases when the same total number of neighbors c may have differing constituent c_d's (the number of neighbors at distance d).

We perform the analysis in two parts: the intermediate levels and the last

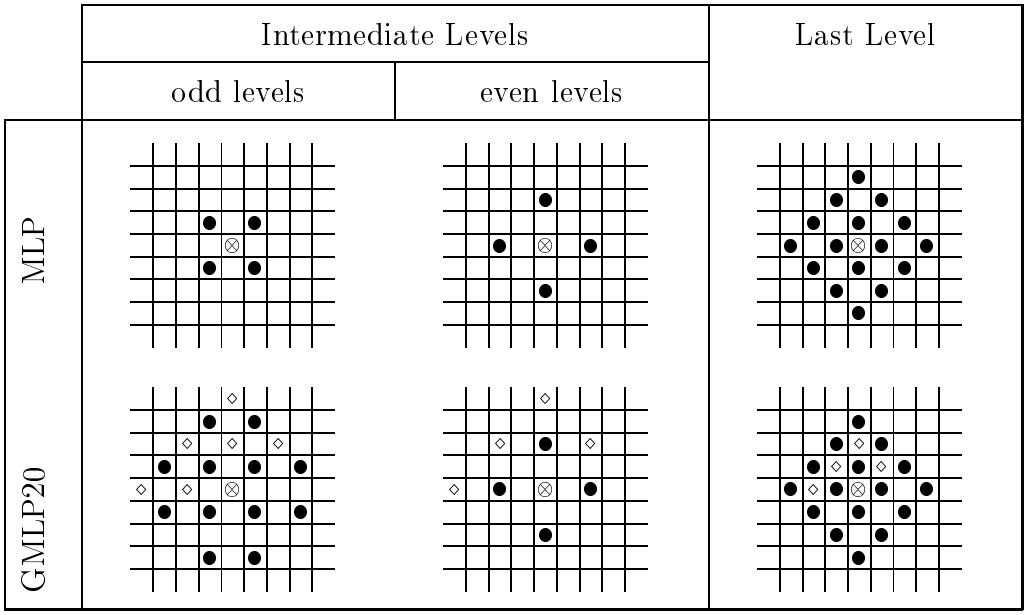


Figure 4.4: The pixel choice used in MLP and GMLP20. The crossed circle $\otimes$ represents the pixel to be predicted. The hollow circles $\circ$ denote the neighbors predicted at the previous levels. The diamonds $\diamond$ denote the neighbors that are predicted at the current level in the usual top-down, left-right sequence.

level:

$$\mathcal{B}_{GMLP20} = sz' + \mathcal{B}_{GMLP20-int} + \mathcal{B}_{GMLP20-las}. \tag{4.3.137}$$

4.3.1 The Intermediate Levels

Summing from an arbitrary level v to m,

$$\mathcal{B}_{GMLP20-int} = \sum_{k=v}^{m} \left\{ \sum_{\text{all } c} n_{k,c} \frac{\sqrt{c_d f(d_k) + c_{2d_k} f(2d_k)}}{c} \right\}. \tag{4.3.138}$$

Substituting $f(d_k) = ud_k$ (2.3.18),

$$\mathcal{B}_{GMLP20-int} = \sum_{k=v}^{m} \left\{ \sum_{\text{all } c} n_{k,c} \frac{\sqrt{ud_k\,(c_d + 2c_{2d_k})}}{c} \right\} \tag{4.3.139}$$

$$= \sqrt{u} \sum_{k=v}^{m} \left\{ \sqrt{d_k} \sum_{\text{all } c} n_{k,c} \frac{\sqrt{c_d + 2c_{2d_k}}}{c} \right\}. \tag{4.3.140}$$

Splitting the summation into odd and even k,

$$\mathcal{B}_{GMLP20-int} = \sqrt{u} \Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{d_k} \sum_{c=1}^{18} n_{k,c} \frac{\sqrt{c_d + 2c_{2d_k}}}{c} \tag{4.3.141}$$

$$+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{d_k} \sum_{c=1}^{8} n_{k,c} \frac{\sqrt{c_d + 2c_{2d_k}}}{c} \Bigg\}]. \tag{4.3.142}$$

Substituting the expressions for $n_{k,c}$ from Table 4.6 and Table 4.7,

$$\begin{aligned}
\mathcal{B}_{GMLP20-int} =& \sqrt{u}\Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{d_k}\Bigg[2\frac{\sqrt{2+2\cdot 3}}{5} + 2\frac{\sqrt{3+2\cdot 4}}{7} + \left(\sqrt{n_k}-4\right)\frac{\sqrt{3+2\cdot 5}}{8} \\
&+ \frac{\sqrt{4+2\cdot 5}}{9} + \frac{\sqrt{4+2\cdot 6}}{10} + \frac{\sqrt{5+2\cdot 5}}{10} \\
&+ \left(\sqrt{n_k}-3\right)\frac{\sqrt{4+2\cdot 7}}{11} + \frac{\sqrt{5+2\cdot 6}}{11} \\
&+ \left(\sqrt{n_k}-3\right)\frac{\sqrt{5+2\cdot 8}}{13} + \frac{\sqrt{6+2\cdot 7}}{13} \\
&+ 2\frac{\sqrt{6+2\cdot 8}}{14} + \left(\sqrt{n_k}-3\right)\frac{\sqrt{6+2\cdot 9}}{15} \\
&+ \left[2\left(\sqrt{n_k}-4\right)+1\right]\frac{\sqrt{6+2\cdot 10}}{16} \\
&+ \left(\sqrt{n_k}-3\right)\frac{\sqrt{6+2\cdot 11}}{17} \\
&+ \left(\sqrt{n_k}-3\right)\left(\sqrt{n_k}-4\right)\frac{\sqrt{6+2\cdot 12}}{18}\Bigg] \\
&+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{d_k}\Bigg[\frac{\sqrt{2+2\cdot 0}}{2} + \left(\sqrt{n_{k-1}}-1\right)\frac{\sqrt{3+2\cdot 1}}{4} + \frac{\sqrt{2+2\cdot 3}}{5} \\
&+ \sqrt{n_{k-1}}\frac{\sqrt{3+2\cdot 2}}{5} + \left(\sqrt{n_{k-1}}-1\right)\frac{\sqrt{3+2\cdot 3}}{6} \\
&+ \frac{\sqrt{4+2\cdot 2}}{6} + 3\left(\sqrt{n_{k-1}}-2\right)\frac{\sqrt{4+2\cdot 3}}{7} \\
&+ \left[\left(\sqrt{n_{k-1}}-1\right)^2 + \left(\sqrt{n_{k-1}}-2\right)^2\right]\frac{\sqrt{4+2\cdot 4}}{8}\Bigg]\Bigg\}
\end{aligned} \tag{4.3.143}$$

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =&\sqrt{u}\Bigg\{\sum_{\substack{k=v\\ k \text{ odd}}}^{m}\sqrt{d_k}\Bigg[\frac{4\sqrt{2}}{5}+\frac{2\sqrt{11}}{7}+\frac{\sqrt{13}}{8}n_k^{\frac{1}{2}}-\frac{1\sqrt{13}}{2}+\frac{\sqrt{14}}{9}+\frac{2}{5}\\ &+\frac{\sqrt{15}}{10}+\frac{3\sqrt{2}}{11}n_k^{\frac{1}{2}}-\frac{9\sqrt{2}}{11}+\frac{\sqrt{17}}{11}+\frac{\sqrt{21}}{13}n_k^{\frac{1}{2}}\\ &-\frac{3\sqrt{21}}{13}+\frac{2\sqrt{5}}{13}+\frac{\sqrt{22}}{7}+\frac{2\sqrt{6}}{15}n_k^{\frac{1}{2}}-\frac{2\sqrt{6}}{5}\\ &+\frac{\sqrt{26}}{8}n_k^{\frac{1}{2}}-\frac{7\sqrt{26}}{16}+\frac{2\sqrt{7}}{17}n_k^{\frac{1}{2}}-\frac{6\sqrt{7}}{17}\\ &+\frac{\sqrt{30}}{18}n_k-\frac{7\sqrt{30}}{18}n_k^{\frac{1}{2}}+\frac{2\sqrt{30}}{3}\Bigg]\\ &+\sum_{\substack{k=v\\ k \text{ even}}}^{m}\sqrt{d_k}\Bigg[\frac{1}{\sqrt{2}}+\frac{\sqrt{5}}{4}n_{k-1}^{\frac{1}{2}}-\frac{\sqrt{5}}{4}+\frac{2\sqrt{2}}{5}+\frac{\sqrt{7}}{5}n_{k-1}^{\frac{1}{2}}\\ &+\frac{1}{2}n_{k-1}^{\frac{1}{2}}-\frac{1}{2}+\frac{\sqrt{2}}{3}+\frac{3\sqrt{10}}{7}n_{k-1}^{\frac{1}{2}}-\frac{6\sqrt{10}}{7}\\ &+\frac{\sqrt{3}}{2}n_{k-1}-\frac{3\sqrt{3}}{2}n_{k-1}^{\frac{1}{2}}+\frac{5\sqrt{3}}{4}\Bigg]\Bigg\}.\end{aligned} \tag{4.3.144}$$

Collecting the n-terms together,

$$\begin{aligned}
\mathcal{B}_{GMLP20-int} =&\sqrt{u}\Bigg\{\sum_{\substack{k=v\\ k \text{ odd}}}^{m}\sqrt{d_k}\Bigg[\quad\left(\frac{\sqrt{30}}{18}\right)n_k\\
&\qquad+\left(\frac{\sqrt{13}}{8}+\frac{3\sqrt{2}}{11}+\frac{\sqrt{21}}{13}+\frac{2\sqrt{6}}{15}+\frac{\sqrt{26}}{8}\right.\\
&\qquad\left.+\frac{2\sqrt{7}}{17}-\frac{7\sqrt{30}}{18}\right)n_k^{\frac{1}{2}}\\
&\qquad+\left(\frac{4\sqrt{2}}{5}+\frac{2\sqrt{11}}{7}-\frac{1\sqrt{13}}{2}+\frac{\sqrt{14}}{9}+\frac{2}{5}+\frac{\sqrt{15}}{10}\right.\\
&\qquad-\frac{9\sqrt{2}}{11}+\frac{\sqrt{17}}{11}-\frac{3\sqrt{21}}{13}+\frac{2\sqrt{5}}{13}+\frac{\sqrt{22}}{7}\\
&\qquad\left.-\frac{2\sqrt{6}}{5}-\frac{7\sqrt{26}}{16}-\frac{6\sqrt{7}}{17}+\frac{2\sqrt{30}}{3}\right)\Bigg]\\
&+\sum_{\substack{k=v\\ k \text{ even}}}^{m}\sqrt{d_k}\Bigg[\quad\left(\frac{\sqrt{3}}{2}\right)n_{k-1}\\
&\qquad+\left(\frac{\sqrt{5}}{4}+\frac{\sqrt{7}}{5}+\frac{1}{2}+\frac{3\sqrt{10}}{7}-\frac{3\sqrt{3}}{2}\right)n_{k-1}^{\frac{1}{2}}\\
&\qquad+\left(\frac{1}{\sqrt{2}}-\frac{\sqrt{5}}{4}+\frac{2\sqrt{2}}{5}-\frac{1}{2}+\frac{\sqrt{2}}{3}\right.\\
&\qquad\left.-\frac{6\sqrt{10}}{7}+\frac{5\sqrt{3}}{4}\right)\Bigg]\Bigg\}. \qquad (4.3.145)
\end{aligned}$$

Replacing the constant expressions in round brackets with $h_1 \ldots h_6$ respectively,

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =&\sqrt{u}\Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{d_k}\Big[\quad h_1 n_k + h_2 n_k^{\frac{1}{2}} + h_3\Big] \\ &+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{d_k}\Big[\quad h_4 n_{k-1} + h_5 n_{k-1}^{\frac{1}{2}} + h_6\Big]\Bigg\}. \end{aligned} \tag{4.3.146}$$

Substituting $n_k = 2^{k-1}s$ (4.1.2), $d_k = 2^{-k/2+1/2}\sqrt{\frac{n}{s}}$ for odd k and $d_k = 2^{-k/2}\sqrt{\frac{n}{s}}$ for even k (4.1.6),

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =&\sqrt{u}\Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{2^{-\frac{k}{2}+\frac{1}{2}}\sqrt{\frac{n}{s}}}\Big[\quad h_1 2^{k-1}s + h_2 2^{\frac{1}{2}(k-1)} s^{\frac{1}{2}} + h_3\Big] \\ &+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{2^{-\frac{k}{2}}\sqrt{\frac{n}{s}}}\Big[\quad h_4 2^{k-2}s + h_5 2^{\frac{1}{2}k-1} s^{\frac{1}{2}} + h_6\Big]\Bigg\}. \end{aligned} \tag{4.3.147}$$

Since $last$ is even, $m = last - 1$ is odd, but we want the two summations to be over the same number of stages. Hence taking out the term when $k = m$ from the summation over odd k's,

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =&\sqrt{u}\Bigg\{ \sum_{\substack{k=v \\ k \text{ odd}}}^{m-1} \sqrt{2^{-\frac{k}{2}+\frac{1}{2}}\sqrt{\frac{n}{s}}}\Big[\quad h_1 2^{k-1}s + h_2 2^{\frac{1}{2}(k-1)} s^{\frac{1}{2}} + h_3\Big] \\ &+ \sqrt{2^{-\frac{m}{2}+\frac{1}{2}}\sqrt{\frac{n}{s}}}\Big[\quad h_1 2^{m-1}s + h_2 2^{\frac{1}{2}(m-1)} s^{\frac{1}{2}} + h_3\Big] \\ &+ \sum_{\substack{k=v \\ k \text{ even}}}^{m-1} \sqrt{2^{-\frac{k}{2}}\sqrt{\frac{n}{s}}}\Big[\quad h_4 2^{k-2}s + h_5 2^{\frac{1}{2}k-1} s^{\frac{1}{2}} + h_6\Big]\Bigg\}. \end{aligned} \tag{4.3.148}$$

Change indexing variable $k \to j$, where odd $k = v + 2j - 2$ and even $k = v + 2j - 1$, assuming m is even and v is odd,

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =& u^{\frac{1}{2}} \left(\frac{n}{s}\right)^{\frac{1}{4}} \left\{ \sum_{j=1}^{\frac{m-v}{2}} \sqrt{2^{-\frac{v+2j-2}{2}+\frac{1}{2}}} \left[h_1 2^{v+2j-3} s + h_2 2^{\frac{1}{2}(v+2j-3)} s^{\frac{1}{2}} + h_3 \right] \right. \\ & \left. + \sum_{j=1}^{\frac{m-v}{2}} \sqrt{2^{-\frac{v+2j-1}{2}}} \left[h_4 2^{v+2j-3} s + h_5 2^{\frac{1}{2}(v+2j-1)-1} s^{\frac{1}{2}} + h_6 \right] \right\} \\ & + \sqrt{2^{-\frac{m}{2}+\frac{1}{2}}} \left[\quad h_1 2^{m-1} s + h_2 2^{\frac{1}{2}(m-1)} s^{\frac{1}{2}} + h_3 \right] \end{aligned} \tag{4.3.149}$$

$$\begin{aligned} =& u^{\frac{1}{2}} \left(\frac{n}{s}\right)^{\frac{1}{4}} \left\{ 2^{\frac{1-v}{4}} \sum_{j=1}^{\frac{m-v}{2}} \left[2^{\frac{1-j}{2}} \left(h_1 2^{v+2j-3} s + h_2 2^{\frac{1}{2}(v+2j-3)} s^{\frac{1}{2}} + h_3 \right) \right. \right. \\ & \left. + 2^{\frac{-j}{2}} \left(h_4 2^{v+2j-3} s + h_5 2^{\frac{1}{2}v+j-\frac{3}{2}} s^{\frac{1}{2}} + h_6 \right) \right] \\ & \left. + \sqrt{2^{-\frac{m}{2}+\frac{1}{2}}} \left[\quad h_1 2^{m-1} s + h_2 2^{\frac{1}{2}(m-1)} s^{\frac{1}{2}} + h_3 \right] \right\}. \end{aligned} \tag{4.3.150}$$

Collecting j-terms,

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =& u^{\frac{1}{2}} \left(\frac{n}{s}\right)^{\frac{1}{4}} \left\{ 2^{\frac{1-v}{4}} \left\{ \sum_{j=1}^{\frac{m-v}{2}} \left[\left(2^{\frac{-5}{2}} h_1 + 2^{-3} h_4 \right) 2^v s 2^{\frac{3}{2}j} \right. \right. \right. \\ & \left. + \left(2^{-1} h_2 + 2^{\frac{-3}{2}} h_5 \right) 2^{\frac{v}{2}} s^{\frac{1}{2}} 2^{\frac{j}{2}} \right. \\ & \left. \left. + \left(2^{\frac{1}{2}} h_3 + h_6 \right) 2^{\frac{-j}{2}} \right] \right. \\ & \left. \left. + \sqrt{2^{-\frac{m}{2}+\frac{1}{2}}} \left[\quad h_1 2^{m-1} s + h_2 2^{\frac{1}{2}(m-1)} s^{\frac{1}{2}} + h_3 \right] \right\}. \end{aligned} \tag{4.3.151}$$

Summing the geometric series $2^{\frac{3}{2}j}$, $2^{\frac{j}{2}}$ and $2^{\frac{-j}{2}}$,

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{\left(2^{\frac{-5}{2}}h_1+2^{-3}h_4\right)2^v s 2^{\frac{3}{2}}\left(\frac{2^{\frac{3}{2}(\frac{m-v}{2})}-1}{2^{\frac{3}{2}}-1}\right)\\ &+\left(2^{-1}h_2+2^{\frac{-3}{2}}h_5\right)2^{\frac{v}{2}}s^{\frac{1}{2}}2^{\frac{1}{2}}\left(\frac{2^{\frac{1}{2}(\frac{m-v}{2})}-1}{2^{\frac{1}{2}}-1}\right)\\ &+\left(2^{\frac{1}{2}}h_3+h_6\right)2^{-\frac{1}{2}}\left(\frac{1-2^{-\frac{1}{2}(\frac{m-v}{2})}}{1-2^{-\frac{1}{2}}}\right)\Bigg\}\\ &+u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}2^{\frac{1}{4}-\frac{m}{4}}\left(\quad h_1 2^{m-1}s+h_2 2^{\frac{1}{2}(m-1)}s^{\frac{1}{2}}+h_3\right)\end{aligned} \tag{4.3.152}$$

$$\begin{aligned}=& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{\left[\left(2^{\frac{-5}{2}}h_1+2^{-3}h_4\right)\left(\frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}}-1}\right)\right]2^v s\left(2^{\frac{3}{2}(\frac{m-v}{2})}-1\right)\\ &+\left[\left(2^{-1}h_2+2^{\frac{-3}{2}}h_5\right)\left(\frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1}\right)\right]2^{\frac{v}{2}}s^{\frac{1}{2}}\left(2^{\frac{1}{2}(\frac{m-v}{2})}-1\right)\\ &+\left[\left(2^{\frac{1}{2}}h_3+h_6\right)\left(\frac{2^{-\frac{1}{2}}}{1-2^{-\frac{1}{2}}}\right)\right]\left(1-2^{-\frac{1}{2}(\frac{m-v}{2})}\right)\Bigg\}\\ &+u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}2^{\frac{1}{4}-\frac{m}{4}}\left(\quad h_1 2^{m-1}s+h_2 2^{\frac{1}{2}(m-1)}s^{\frac{1}{2}}+h_3\right).\end{aligned} \tag{4.3.153}$$

Replacing constant expressions in $[\ldots]$ again,

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{h_7 2^v s\left(2^{\frac{3}{2}(\frac{m-v}{2})}-1\right)\\ &+h_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\left(2^{\frac{1}{2}(\frac{m-v}{2})}-1\right)\\ &+h_9\left(1-2^{-\frac{1}{2}(\frac{m-v}{2})}\right)\Bigg\}\\ &+u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1}{4}-\frac{m}{4}}\Bigg(\quad h_1 2^{m-1}s+h_2 2^{\frac{1}{2}(m-1)}s^{\frac{1}{2}}+h_3\Bigg)\end{aligned} \tag{4.3.154}$$

$$\begin{aligned}\mathcal{B}_{GMLP20-int} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1-v}{4}}\Bigg\{h_7 2^v s 2^{\frac{3}{4}(m-v)}-h_7 2^v s\\ &+h_8 2^{\frac{v}{2}} s^{\frac{1}{2}} 2^{\frac{1}{4}(m-v)}-h_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\\ &+h_9-h_9 2^{-\frac{1}{4}(m-v)}\Bigg\}\\ &+u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1}{4}-\frac{m}{4}}\Bigg(\quad h_1 2^{m-1}s+h_2 2^{\frac{1}{2}(m-1)}s^{\frac{1}{2}}+h_3\Bigg)\end{aligned} \tag{4.3.155}$$

$$\begin{aligned}=& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\Bigg\{2^{\frac{1}{4}}h_7 s 2^{\frac{3}{4}m}+2^{\frac{1}{4}}h_8 s^{\frac{1}{2}}2^{\frac{1}{4}m}\\ &+\left(h_9-h_7 2^v s-h_8 2^{\frac{v}{2}}s^{\frac{1}{2}}\right)2^{\frac{1-v}{4}}-h_9 2^{\frac{1}{4}(1-m)}\Bigg\}\\ &+u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1}{4}-\frac{m}{4}}\Bigg(\quad h_1 2^{m-1}s+h_2 2^{\frac{1}{2}(m-1)}s^{\frac{1}{2}}+h_3\Bigg).\end{aligned} \tag{4.3.156}$$

Substituting $m = \log_2\left(\frac{n}{s}\right) - 1$,

$$\begin{aligned}
\mathcal{B}_{GMLP20-int} =& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\left\{2^{\frac{1}{4}}h_7 s 2^{\frac{3}{4}\left(\log_2\left(\frac{n}{s}\right)-1\right)} + 2^{\frac{1}{4}}h_8 s^{\frac{1}{2}} 2^{\frac{1}{4}\left(\log_2\left(\frac{n}{s}\right)-1\right)}\right. \\
&\left. + \left(h_9 - h_7 2^v s - h_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} - h_9 2^{\frac{1}{4}\left[1-\left(\log_2\left(\frac{n}{s}\right)-1\right)\right]}\right\} \\
&+ u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1}{4}-\frac{1}{4}\left(\log_2\left(\frac{n}{s}\right)-1\right)}\left(h_1 2^{\log_2\left(\frac{n}{s}\right)-1-1} s\right. \\
&\left. + h_2 2^{\frac{1}{2}\left(\log_2\left(\frac{n}{s}\right)-1-1\right)} s^{\frac{1}{2}} + h_3\right) \qquad (4.3.157) \\
=& u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\left\{2^{-\frac{1}{2}}h_7 s\left(\frac{n}{s}\right)^{\frac{3}{4}} + h_8 s^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}}\right. \\
&\left. + \left(h_9 - h_7 2^v s - h_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} - 2^{\frac{1}{2}} h_9 \left(\frac{n}{s}\right)^{-\frac{1}{4}}\right\} \qquad (4.3.158) \\
&+ u^{\frac{1}{2}}\left(\frac{n}{s}\right)^{\frac{1}{4}} 2^{\frac{1}{2}}\left(\frac{n}{s}\right)^{-\frac{1}{4}}\left(h_1 2^{-2}\left(\frac{n}{s}\right) s + h_2 2^{-1}\left(\frac{n}{s}\right)^{\frac{1}{2}} s^{\frac{1}{2}} + h_3\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{GMLP20-int} =& u^{\frac{1}{2}}\left\{2^{-\frac{1}{2}}h_7 n + h_8 n^{\frac{1}{2}} + \left(h_9 - h_7 2^v s - h_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} n^{\frac{1}{4}} s^{-\frac{1}{4}} - 2^{\frac{1}{2}} h_9\right\} \\
&+ u^{\frac{1}{2}}\left(h_1 2^{-\frac{3}{2}} n + h_2 2^{-\frac{1}{2}} n^{\frac{1}{2}} + 2^{\frac{1}{2}} h_3\right) \qquad (4.3.159) \\
=& u^{\frac{1}{2}}\left\{\left(2^{-\frac{1}{2}}h_7 + 2^{-\frac{3}{2}}h_1\right) n + \left(h_8 + 2^{-\frac{1}{2}}h_2\right) n^{\frac{1}{2}}\right. \\
&\left. + \left(h_9 - h_7 2^v s - h_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} n^{\frac{1}{4}} s^{-\frac{1}{4}} + 2^{\frac{1}{2}}\left(h_3 - h_9\right)\right\} \qquad (4.3.160)
\end{aligned}$$

where,

$$h_1 = \frac{\sqrt{30}}{18} \qquad \approx 0.3043 \tag{4.3.161}$$

$$h_2 = \frac{\sqrt{13}}{8} + \frac{3\sqrt{2}}{11} + \frac{\sqrt{21}}{13} + \frac{2\sqrt{6}}{15} + \frac{\sqrt{26}}{8} + \frac{2\sqrt{7}}{17} - \frac{7\sqrt{30}}{18} \tag{4.3.162}$$

$$h_3 = \frac{4\sqrt{2}}{5} + \frac{2\sqrt{11}}{7} - \frac{1\sqrt{13}}{2} + \frac{\sqrt{14}}{9} + \frac{2}{5} + \frac{\sqrt{15}}{10} - \frac{9\sqrt{2}}{11} + \frac{\sqrt{17}}{11} - \frac{3\sqrt{21}}{13} + \frac{2\sqrt{5}}{13} + \frac{\sqrt{22}}{7} - \frac{2\sqrt{6}}{5} - \frac{7\sqrt{26}}{16} - \frac{6\sqrt{7}}{17} + \frac{2\sqrt{30}}{3} \tag{4.3.163}$$

$$h_4 = \frac{\sqrt{3}}{2} \qquad \approx 0.8660 \tag{4.3.164}$$

$$h_5 = \frac{\sqrt{5}}{4} + \frac{\sqrt{7}}{5} + \frac{1}{2} + \frac{3\sqrt{10}}{7} - \frac{3\sqrt{3}}{2} \tag{4.3.165}$$

$$h_6 = \frac{1}{\sqrt{2}} - \frac{\sqrt{5}}{4} + \frac{2\sqrt{2}}{5} - \frac{1}{2} + \frac{\sqrt{2}}{3} - \frac{6\sqrt{10}}{7} + \frac{5\sqrt{3}}{4} \tag{4.3.166}$$

$$h_7 = \left(2^{\frac{-5}{2}} h_1 + 2^{-3} h_4\right) \left(\frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}} - 1}\right) \qquad \approx 0.2507 \tag{4.3.167}$$

$$h_8 = \left(2^{-1} h_2 + 2^{\frac{-3}{2}} h_5\right) \left(\frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}} - 1}\right) \tag{4.3.168}$$

$$h_9 = \left(2^{\frac{1}{2}} h_3 + h_6\right) \left(\frac{2^{-\frac{1}{2}}}{1 - 2^{-\frac{1}{2}}}\right). \tag{4.3.169}$$

4.3.2 The Last Level

$$\mathcal{B}_{GMLP20-las} = \sum_{c=1}^{20} n_{last,c} \frac{\sqrt{\sum_d c_d f(d)}}{c}. \tag{4.3.170}$$

c	c_d	c_{2d}	$n_{k,c}$ for odd k
5	2	3	2
7	3	4	2
8	3	5	$\sqrt{n_k} - 4$
9	4	5	1
10	4	6	1
10	5	5	1
11	4	7	$\sqrt{n_k} - 3$
11	5	6	1
13	5	8	$\sqrt{n_k} - 3$
13	6	7	1
14	6	8	2
15	6	9	$(\sqrt{n_k} - 4) + 1$
16	6	10	$2(\sqrt{n_k} - 4) + 1$
17	6	11	$\sqrt{n_k} - 3$
18	6	12	$(\sqrt{n_k} - 3)(\sqrt{n_k} - 4)$
otherwise			0

Table 4.6: $n_{k,c}$ for GMLP18 odd intermediate levels.

c	c_d	c_{2d}	$n_{k,c}$ for even k
2	2	0	1
4	3	1	$\sqrt{n_{k-1}} - 1$
5	2	3	1
5	3	2	$(\sqrt{n_{k-1}} - 1) + 1$
6	3	3	$(\sqrt{n_{k-1}} - 2) + 1$
6	4	2	1
7	4	3	$3(\sqrt{n_{k-1}} - 2)$
8	4	4	$(\sqrt{n_{k-1}} - 1)^2 + (\sqrt{n_{k-1}} - 2)^2$
otherwise			0

Table 4.7: $n_{k,c}$ for GMLP18 even intermediate levels.

$c = \sum_d c_d$	c_1	c_2	c_3	$n_{k,c}$ for odd last level k
6	2	0	4	1
9	2	3	4	1
9	3	1	5	1
10	3	2	5	2
11	3	3	5	1
10	3	1	6	1
11	3	2	6	1
12	3	3	6	1
13	3	4	6	1
11	3	1	7	$\left(\frac{\sqrt{n}-6}{2}\right)$
12	3	2	7	$\left(\frac{\sqrt{n}-6}{2}\right)+1$
13	3	3	7	$\left(\frac{\sqrt{n}-6}{2}\right)$
14	3	4	7	$\left(\frac{\sqrt{n}-6}{2}\right)+1$
15	4	3	8	2
16	4	4	8	2
16	4	3	9	$2\left(\frac{\sqrt{n}-6}{2}\right)$
17	4	4	9	$2\left(\frac{\sqrt{n}-6}{2}\right)$
18	4	4	10	2
19	4	4	11	$4\left(\frac{\sqrt{n}-6}{2}\right)$
20	4	4	12	$(\sqrt{n}-6)\left(\frac{\sqrt{n}-6}{2}\right)$
otherwise				0

Table 4.8: $n_{k,c}$ for GMLP18 last level. Our assumptions on n and s will make the last level even.

Substituting $f(d) = ud$ (2.3.18),

$$\mathcal{B}_{GMLP20-las} = \sqrt{u} \sum_{c=1}^{20} n_{last,c} \frac{\sqrt{\sum_d c_d d}}{c}. \tag{4.3.171}$$

Substituting the expressions for $n_{k,c}$ from Table 4.2 ,

$$\begin{aligned}
\mathcal{B}_{GMLP20-las} = \sqrt{u} \Bigg\{ & \frac{\sqrt{2+2\cdot 0+3\cdot 4}}{6} + \frac{\sqrt{2+2\cdot 3+3\cdot 4}}{9} + \frac{\sqrt{3+2\cdot 1+3\cdot 5}}{9} \\
& + 2\frac{\sqrt{3+2\cdot 2+3\cdot 5}}{10} + \frac{\sqrt{3+2\cdot 3+3\cdot 5}}{11} + \frac{\sqrt{3+2\cdot 1+3\cdot 6}}{10} \\
& + \frac{\sqrt{3+2\cdot 2+3\cdot 6}}{11} + \frac{\sqrt{3+2\cdot 3+3\cdot 6}}{12} + \frac{\sqrt{3+2\cdot 4+3\cdot 6}}{13} \\
& + \left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{3+2\cdot 1+3\cdot 7}}{11} \\
& + \left(\frac{\sqrt{n}-6}{2}+1\right) \frac{\sqrt{3+2\cdot 2+3\cdot 7}}{12} \\
& + \left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{3+2\cdot 3+3\cdot 7}}{13} \\
& + \left(\frac{\sqrt{n}-6}{2}+1\right) \frac{\sqrt{3+2\cdot 4+3\cdot 7}}{14} \\
& + 2\frac{\sqrt{4+2\cdot 3+3\cdot 8}}{15} + 2\frac{\sqrt{4+2\cdot 4+3\cdot 8}}{16} \\
& + 2\left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{4+2\cdot 3+3\cdot 9}}{16} \\
& + 2\left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{4+2\cdot 4+3\cdot 9}}{17} \\
& + 2\frac{\sqrt{4+2\cdot 4+3\cdot 10}}{18} + 4\left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{4+2\cdot 4+3\cdot 11}}{19} \\
& + \frac{1}{2}\left(\sqrt{n}-6\right)^2 \frac{\sqrt{4+2\cdot 4+3\cdot 12}}{20} \Bigg\}
\end{aligned} \tag{4.3.172}$$

$$\begin{aligned}\mathcal{B}_{GMLP20-las} =&\sqrt{u}\Bigg\{\frac{\sqrt{14}}{6}+\frac{2\sqrt{5}}{9}+\frac{2\sqrt{5}}{9}+2\frac{\sqrt{22}}{10}+\frac{2\sqrt{6}}{11}+\frac{\sqrt{23}}{10}\\&+\frac{5}{11}+\frac{\sqrt{3}}{4}+\frac{\sqrt{29}}{13}+\left(\frac{\sqrt{n}-6}{2}\right)\frac{2\sqrt{6}}{11}+\left(\frac{\sqrt{n}-6}{2}+1\right)\frac{\sqrt{7}}{6}\\&+\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{30}}{13}+\left(\frac{\sqrt{n}-6}{2}+1\right)\frac{2\sqrt{2}}{7}+2\frac{\sqrt{34}}{15}+\frac{3}{4}\\&+\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{37}}{8}+2\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{39}}{17}\\&+\frac{\sqrt{42}}{9}+4\left(\frac{\sqrt{n}-6}{2}\right)\frac{\sqrt{43}}{19}\\&+\left(n-12\sqrt{n}+36\right)\frac{\sqrt{3}}{10}\Bigg\}.\end{aligned}\tag{4.3.173}$$

Collecting n-terms,

$$\begin{aligned}\mathcal{B}_{GMLP20-las} =&\sqrt{u}\Bigg\{\quad\left(\frac{\sqrt{3}}{10}\right)n+\Bigg(\frac{\sqrt{6}}{11}+\frac{\sqrt{7}}{12}+\frac{\sqrt{30}}{26}+\frac{\sqrt{2}}{7}+\frac{\sqrt{37}}{16}\\&+\frac{\sqrt{39}}{17}+2\frac{\sqrt{43}}{19}-6\frac{\sqrt{3}}{5}\Bigg)\sqrt{n}+\Bigg(\frac{\sqrt{14}}{6}+\frac{2\sqrt{5}}{9}\\&+\frac{2\sqrt{5}}{9}+2\frac{\sqrt{22}}{10}+\frac{2\sqrt{6}}{11}+\frac{\sqrt{23}}{10}+\frac{5}{11}+\frac{\sqrt{3}}{4}\\&+\frac{\sqrt{29}}{13}-3\frac{2\sqrt{6}}{11}-2\frac{\sqrt{7}}{6}-3\frac{\sqrt{30}}{13}-2\frac{2\sqrt{2}}{7}+2\frac{\sqrt{34}}{15}\\&+\frac{3}{4}-3\frac{\sqrt{37}}{8}-6\frac{\sqrt{39}}{17}+\frac{\sqrt{42}}{9}-12\frac{\sqrt{43}}{19}+18\frac{\sqrt{3}}{5}\Bigg)\Bigg\}.\end{aligned}\tag{4.3.174}$$

Replacing constant expressions in $(\dots)$ with h_{10}, h_{11} and h_{12},

$$\mathcal{B}_{GMLP20-las} =\sqrt{u}\Big[h_{10}n+h_{11}\sqrt{n}+h_{12}\Big]\tag{4.3.175}$$

where the constants,

$$h_{10} = \frac{\sqrt{3}}{10} \approx 0.1732 \tag{4.3.176}$$

$$\begin{aligned} h_{11} =& \frac{\sqrt{6}}{11} + \frac{\sqrt{7}}{12} + \frac{\sqrt{30}}{26} + \frac{\sqrt{2}}{7} \\ &+ \frac{\sqrt{37}}{16} + \frac{\sqrt{39}}{17} + 2\frac{\sqrt{43}}{19} - 6\frac{\sqrt{3}}{5} \end{aligned} \tag{4.3.177}$$

$$\begin{aligned} h_{12} =& \frac{\sqrt{14}}{6} + \frac{2\sqrt{5}}{9} + \frac{2\sqrt{5}}{9} + 2\frac{\sqrt{22}}{10} \\ &+ \frac{2\sqrt{6}}{11} + \frac{\sqrt{23}}{10} + \frac{5}{11} + \frac{\sqrt{3}}{4} \\ &+ \frac{\sqrt{29}}{13} - 3\frac{2\sqrt{6}}{11} - 2\frac{\sqrt{7}}{6} - 3\frac{\sqrt{30}}{13} \\ &- 2\frac{2\sqrt{2}}{7} + 2\frac{\sqrt{34}}{15} + \frac{3}{4} - 3\frac{\sqrt{37}}{8} \\ &- 6\frac{\sqrt{39}}{17} + \frac{\sqrt{42}}{9} - 12\frac{\sqrt{43}}{19} + 18\frac{\sqrt{3}}{5}. \end{aligned} \tag{4.3.178}$$

4.3.3 The Badness of GMLP20

Combining Equations (4.3.160) and (4.3.175),

$$\begin{aligned} \mathcal{B}_{GMLP20} =& sz' \\ &+ u^{\frac{1}{2}} \Bigg[\left(2^{-\frac{3}{2}} h_1 + 2^{-\frac{1}{2}} h_7\right) n + \left(2^{-\frac{1}{2}} h_2 + h_8\right) n^{\frac{1}{2}} \\ &\quad + \left(h_9 - h_7 2^v s - h_8 2^{\frac{v}{2}} s^{\frac{1}{2}}\right) 2^{\frac{1-v}{4}} s^{-\frac{1}{4}} n^{\frac{1}{4}} + 2^{\frac{1}{2}} \left(h_3 - h_9\right) \Bigg] \\ &+ u^{\frac{1}{2}} \Big[h_{10} n + h_{11} n^{\frac{1}{2}} + h_{12} \Big]. \end{aligned} \tag{4.3.179}$$

Substituting $v = 1$,

$$\begin{aligned}\mathcal{B}_{GMLP20} =& sz' \\ &+ u^{\frac{1}{2}}\Bigg[\left(2^{-\frac{3}{2}}h_1 + 2^{-\frac{1}{2}}h_7\right)n + \left(2^{-\frac{1}{2}}h_2 + h_8\right)n^{\frac{1}{2}} \\ &\qquad + \left(h_9 - 2h_7 s - 2^{\frac{1}{2}}h_8 s^{\frac{1}{2}}\right)s^{-\frac{1}{4}}n^{\frac{1}{4}} + 2^{\frac{1}{2}}\left(h_3 - h_9\right)\Bigg] \\ &+ u^{\frac{1}{2}}\Big[h_{10}n + h_{11}n^{\frac{1}{2}} + h_{12}\Big]. \end{aligned} \tag{4.3.180}$$

Collecting n-terms,

$$\begin{aligned}\mathcal{B}_{GMLP20} =& sz' \\ &+ u^{\frac{1}{2}}\Bigg[\left(2^{-\frac{3}{2}}h_1 + 2^{-\frac{1}{2}}h_7 + h_{10}\right)n \\ &\qquad + \left(2^{-\frac{1}{2}}h_2 + h_8 + h_{11}\right)n^{\frac{1}{2}} \\ &\qquad + \left(h_9 - 2h_7 s - 2^{\frac{1}{2}}h_8 s^{\frac{1}{2}}\right)s^{-\frac{1}{4}}n^{\frac{1}{4}} \\ &\qquad + \left(2^{\frac{1}{2}}h_3 - 2^{\frac{1}{2}}h_9 + h_{12}\right)\Bigg]. \end{aligned} \tag{4.3.181}$$

Substituting $u = \frac{w}{\sqrt{n}}$ (2.3.20),

$$\begin{aligned}\mathcal{B}_{GMLP20} =& sz' \\ &+ \left(2^{-\frac{3}{2}}h_1 + 2^{-\frac{1}{2}}h_7 + h_{10}\right)w^{\frac{1}{2}}n^{\frac{3}{4}} \\ &+ \left(2^{-\frac{1}{2}}h_2 + h_8 + h_{11}\right)w^{\frac{1}{2}}n^{\frac{1}{4}} \\ &+ \left(h_9 - 2h_7 s - 2^{\frac{1}{2}}h_8 s^{\frac{1}{2}}\right)s^{-\frac{1}{4}}w^{\frac{1}{2}} \\ &+ \left(2^{\frac{1}{2}}h_3 - 2^{\frac{1}{2}}h_9 + h_{12}\right)w^{\frac{1}{2}}n^{-\frac{1}{4}}. \end{aligned} \tag{4.3.182}$$

The dominant term will evaluate to $0.458\sqrt{w}n^{\frac{3}{4}}$ which is slightly worse than raster scan's $0.441\sqrt{w}n^{\frac{3}{4}}$ at $l = 3$ and better than raster scan's $0.527\sqrt{w}n^{\frac{3}{4}}$ at $l = 2$. This seems to suggest that GMLP20 is probably better than most raster scan orderings using pixel choices with neighbors atmost distance 2 away.

In the next chapter, we take a respite from this analysis and investigate the relationship between pixel choice and the weights used in prediction.

Chapter 5

The Problem of Weights

Nothing yet has been said about the weights used in prediction functions. In this chapter, we investigate its relationship with pixel choice and how it affects our badness measure. Recall that pixel choice is closely related to pixel ordering in a mutually constraining way; consequently, the problem of weights indirectly affects pixel ordering too. A modified badness measure is derived and proposed to take into account the weights.

5.1 The Problem

Our previously proposed measure $\mathcal{B}$ seeks to evaluate a pixel ordering and pixel choice combination by quantifying *which* neighbors are used and *how far* these neighbors are. One wonders whether and how the weights in a prediction function affect pixel choice. It is surely preferable to study the pixel ordering problem independent of the prediction function (and hence the weights), but

neighbors whose weights are very closed to zero are effectively not neighbors in the pixel choice at all. This means that a measure for the pixel ordering problem can ignore those neighbors.

Looking at the pixel ordering problem this way, a prediction in an n-pixel image can be considered to have a pixel choice of $n-1$ neighbors each of which has a normalized weight. In the case of our original badness measure $\mathcal{B}$, these weights take discrete values from the set $\{0, \frac{1}{c}\}$ (where c is the total number of neighbors actually used). Obviously, these discrete weights were a simplification and may render our measure inadequate for analyzing predictions that do not use the averaging function as the predictor: we need to somehow take the weights of the predictor into account.

Instead of proposing a measure which takes a set of weights as parameters, we would like to study pixel ordering under the condition of optimal weights; we propose a measure to quantify how bad a given pixel ordering using the best possible weights. There are two reasons for this approach. Firstly, this helps us to focus on the pixel ordering problem (and not the best predictor problem). Secondly, optimal weights enforce a kind lower bound, since a pixel ordering using non-optimal weights cannot be better than the same one using optimal weights.

The next two sections will show how we derive this new optimally weighted badness measure.

5.2 Finding the Optimal Weights

Let $X_1, X_2, \ldots, X_c$ be the random variables for the pixel values of the c neighbors used in an arbitrary prediction and $w_1, w_2, \ldots, w_c$ their corresponding normalized weights. The random variable $\hat{Y}$ for the predicted value is then,

$$\hat{Y} = w_1 X_1 + w_2 X_2 + \ldots + w_c X_c. \tag{5.2.1}$$

If Y is the actual value of the pixel, the error will be given by,

$$\hat{Y} - Y = w_1 X_1 + w_2 X_2 + \ldots + w_c X_c - Y. \tag{5.2.2}$$

Since $\sum_i^c w_i = 1$, we distribute Y according to w_i,

$$\hat{Y} - Y = w_1(X_1 - Y) + w_2(X_2 - Y) + \ldots + w_c(X_c - Y). \tag{5.2.3}$$

But we are interested in variances, so,

$$\mathrm{Var}(\hat{Y} - Y) = \mathrm{Var}\left[w_1(X_1 - Y) + w_2(X_2 - Y) + \ldots + w_c(X_c - Y)\right]. \tag{5.2.4}$$

Assuming $(X_i - Y)$ are independent,

$$\mathrm{Var}(\hat{Y} - Y) = w_1^2 \mathrm{Var}(X_1 - Y) + w_2^2 \mathrm{Var}(X_2 - Y) + \ldots + w_c^2 \mathrm{Var}(X_c - Y). \tag{5.2.5}$$

Estimating $\text{Var}(X_i - Y)$ with $f(d_i)$, where d_i is the distance of the pixel corresponding X_i from the predicted pixel,

$$\text{Var}(\hat{Y} - Y) = w_1^2 f(d_1) + w_2^2 f(d_2) + \ldots + w_c^2 f(d_c). \tag{5.2.6}$$

Our goal is compression and that means that we want to minimize $\text{Var}(\hat{Y} - Y)$ (see Section 2.2); hence we use Lagrange's Method to obtain the optimal weights. Let

$$F(\vec{w}) = w_1^2 f(d_1) + w_2^2 f(d_2) + \ldots + w_c^2 f(d_c) \tag{5.2.7}$$

$$G(\vec{w}) = w_1 + w_2 + \ldots + w_c = 1 \tag{5.2.8}$$

$$\begin{aligned} u &= F(\vec{w}) + \lambda G(\vec{w}) \\ &= w_1^2 f(d_1) + w_2^2 f(d_2) + \ldots + w_c^2 f(d_c) + \lambda \left(w_1 + w_2 + \ldots + w_c\right) \end{aligned} \tag{5.2.9}$$

where $F(\vec{w})$ is the expression to be minimized under the constraint $G(\vec{w})$ and λ is the Lagrange multiplier.

Differentiating u partially with respect to each w_i,

$$\frac{\partial u}{\partial w_i} = 2f(d_i)w_i + \lambda. \tag{5.2.10}$$

At minima, each $\frac{\partial u}{\partial w_i}$ must be 0,

$$2f(d_i)w_i + \lambda = 0 \tag{5.2.11}$$

$$w_i = -\frac{\lambda}{2f(d_i)}. \tag{5.2.12}$$

Substituting 5.2.12 into $G(\vec{w})$ (5.2.8) to solve for λ,

$$\sum_{j=1}^{c} -\frac{\lambda}{2f(d_j)} = 1 \tag{5.2.13}$$

$$\frac{\lambda}{2}\sum_{j=1}^{c} \frac{1}{f(d_j)} = -1 \tag{5.2.14}$$

$$\lambda = -\frac{2}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}. \tag{5.2.15}$$

Back-substituting 5.2.15 into 5.2.12 for optimal weights,

$$w_i = \left(-\frac{1}{2f(d_i)}\right)\left(-\frac{2}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}\right) \tag{5.2.16}$$

$$= \frac{1}{f(d_i)\sum_{j=1}^{c} \frac{1}{f(d_j)}}. \tag{5.2.17}$$

5.3 The Optimal Weights Badness Measure

Since $F(\vec{w}) = \mathrm{Var}(\hat{Y} - Y)$, we substitute the optimal weights into $F(\vec{w})$ to get the optimal variance of the errors of a prediction,

$$F(\vec{w}_{\text{optimal}}) = \sum_{i=1}^{c}\left\{\left[\frac{1}{f(d_i)\sum_{j=1}^{c} \frac{1}{f(d_j)}}\right]^2 f(d_i)\right\} \tag{5.3.18}$$

$$= \sum_{i=1}^{c} \frac{f(d_i)}{[f(d_i)]^2\left[\sum_{j=1}^{c} \frac{1}{f(d_j)}\right]^2}. \tag{5.3.19}$$

But $\left[\sum_{j=1}^{c} \frac{1}{f(d_j)}\right]^2$ is constant relative to i,

$$F(\vec{w}_{\text{optimal}}) = \frac{1}{\left[\sum_{j=1}^{c} \frac{1}{f(d_j)}\right]^2} \sum_{i=1}^{c} \frac{f(d_i)}{\left[f(d_i)\right]^2} \tag{5.3.20}$$

$$= \frac{1}{\left[\sum_{j=1}^{c} \frac{1}{f(d_j)}\right]^2} \sum_{i=1}^{c} \frac{1}{f(d_i)} \tag{5.3.21}$$

$$= \frac{1}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}. \tag{5.3.22}$$

We are now ready to propose a badness measure that uses the optimal weights for the given pixel ordering and pixel choice combination. Recall that the badness measure is the sum of the estimates of the standard deviation of the errors of each prediction. $F(\vec{w}_{\text{optimal}})$ estimates the variance of the errors of each prediction; hence the optimal weights badness measure is given by,

$$\mathcal{B}' = \sum_{\text{all pixels}} \begin{cases} z' & \text{no prediction,} \\ \sqrt{F(\vec{w}_{\text{optimal.}})} & \text{otherwise} \end{cases} \tag{5.3.23}$$

$$= \sum_{\text{all pixels}} \begin{cases} z' & \text{no prediction,} \\ \sqrt{\frac{1}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}} & \text{otherwise.} \end{cases} \tag{5.3.24}$$

where j indexes each of the c neighbors used for prediction and d_j is the distance of neighbor j from the predicted pixel.

5.4 An Optimal Pixel Ordering/Pixel Choice?

Since we can interpret whether a pixel should become a neighbor in a prediction based on how small the weight for that pixel is, we ask if there could be some optimal sequence and choice of neighbors?

From Section 5.2 Equation 5.2.17, the expression for optimal weight is,

$$w_i = \frac{1}{f(d_i)\sum_{j=1}^{c}\frac{1}{f(d_j)}} \tag{5.4.25}$$

$$= \frac{1}{f(d_i)\left[\frac{1}{f(d_1)}+\frac{1}{f(d_2)}+\cdots+\frac{1}{f(d_c)}\right]} \tag{5.4.26}$$

$$= \frac{1}{f(d_i)\left[\frac{f(d_2)f(d_3)\ldots f(d_c)+f(d_1)f(d_3)\ldots f(d_c)+\cdots+f(d_1)f(d_2)f(d_3)\ldots f(d_{c-1})}{f(d_1)f(d_2)f(d_3)\ldots f(d_c)}\right]} \tag{5.4.27}$$

$$= \frac{f(d_1)f(d_2)f(d_3)\ldots f(d_c)}{f(d_i)\left[\begin{array}{c} f(d_2)f(d_3)\ldots f(d_c)+f(d_1)f(d_3)\ldots f(d_c)+\cdots \\ +f(d_1)f(d_2)f(d_3)\ldots f(d_{c-1}) \end{array}\right]} \tag{5.4.28}$$

$$= \frac{f(d_1)f(d_2)f(d_3)\ldots f(d_{i-1})f(d_{i+1})\ldots f(d_c)}{\left[\begin{array}{c} f(d_2)f(d_3)\ldots f(d_c)+f(d_1)f(d_3)\ldots f(d_c)+\cdots \\ +f(d_1)f(d_2)f(d_3)\ldots f(d_{c-1}) \end{array}\right]}. \tag{5.4.29}$$

Note that the denominator is constant over all i; hence, the value of w_i depends on the numerator only. Since $f(d)$ has positive gradient over all d's for most images, w_i can only be small if the distance of neighbor i is relatively larger than the distances of the other neighbors. This confirms that near neighbors are better predictors; however, this result is as good as the model for $f(d)$.

Chapter 6

Analysis of Raster Scan with Optimal Weights

Using the badness measure with optimal weights as defined in Equation 5.3.24,

$$\mathcal{B}'_{rs} = sz' + \sum_{p=1}^{n-s} \sqrt{\frac{1}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}} \tag{6.0.1}$$

where z' is the cost of encoding the unpredictable pixels, c is the number of neighbors used, n is the number of pixels the image has and d_j the distance of the j-th neighbor. Partitioning all neighbors according to their distance,

$$\mathcal{B}'_{rs} = sz' + \sum_{p=1}^{n-s} \sqrt{\frac{1}{\sum_{d=1}^{l} \frac{c_d}{f(d)}}}. \tag{6.0.2}$$

where c_d is the number of neighbors at distance d. For raster scan, $c_d \leq 2d$; using the maximum c_d,

$$\mathcal{B}'_{rs} = sz' + \sum_{p=1}^{n-s} \frac{1}{\sqrt{\sum_{d=1}^{l} \frac{2d}{f(d)}}}. \tag{6.0.3}$$

6.1 Analysis Using Linear Model for $f(d)$

$$\mathcal{B}'_{rs,lin} = sz' + \sum_{p=1}^{n-s} \frac{1}{\sqrt{\sum_{d=1}^{l} \frac{2d}{ud}}} \tag{6.1.4}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{1}{\sqrt{\sum_{d=1}^{l} \frac{2}{u}}} \tag{6.1.5}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{1}{\sqrt{\frac{2l}{u}}} \tag{6.1.6}$$

$$= sz' + (n-s)\sqrt{\frac{u}{2l}}. \tag{6.1.7}$$

Substituting $u = \frac{w}{\sqrt{n}}$ (2.3.20)

$$\mathcal{B}'_{rs,lin} = sz' + (n-s)\sqrt{\frac{\left(\frac{w}{\sqrt{n}}\right)}{2l}} \tag{6.1.8}$$

$$= sz' + (n-s)n^{-\frac{1}{4}}\sqrt{\frac{w}{2l}}. \tag{6.1.9}$$

6.2 Analysis Using Sublinear Model for $f(d)$

$$\mathcal{B}'_{rs,sub} = sz' + \sum_{p=1}^{n-s} \frac{1}{\sqrt{\sum_{d=1}^{l} \frac{2d}{u\sqrt{d}}}} \tag{6.2.10}$$

$$= sz' + (n-s)\sqrt{\frac{u}{\sum_{d=1}^{l} 2\sqrt{d}}}. \tag{6.2.11}$$

Using the approximation $\sum_{i=1}^{l} i^{\frac{1}{2}} \approx \int_{0.5}^{l+0.5} i^{\frac{1}{2}} di = \frac{2}{3}(l+0.5)^{\frac{3}{2}} - \frac{2}{3}(0.5)^{\frac{3}{2}}$,

$$\mathcal{B}'_{rs,sub} \sim sz' + (n-s)\sqrt{\frac{u}{2\left[\frac{2}{3}(l+0.5)^{\frac{3}{2}} - \frac{2}{3}(0.5)^{\frac{3}{2}}\right]}} \tag{6.2.12}$$

$$= sz' + (n-s)\sqrt{\frac{u}{\frac{4}{3}\left[(l+0.5)^{\frac{3}{2}} - (0.5)^{\frac{3}{2}}\right]}} \tag{6.2.13}$$

$$= sz' + (n-s)\sqrt{\frac{3u}{4\left[(l+0.5)^{\frac{3}{2}} - (0.5)^{\frac{3}{2}}\right]}}. \tag{6.2.14}$$

Substituting $u = \frac{w}{\sqrt[4]{n}}$ (2.3.30)

$$\mathcal{B}'_{rs,sub} \sim sz' + (n-s)\sqrt{\frac{3\frac{w}{\sqrt[4]{n}}}{4\left[(l+0.5)^{\frac{3}{2}} - (0.5)^{\frac{3}{2}}\right]}} \tag{6.2.15}$$

$$= sz' + (n-s)n^{-\frac{1}{8}}\sqrt{\frac{3w}{4\left[(l+0.5)^{\frac{3}{2}} - (0.5)^{\frac{3}{2}}\right]}}. \tag{6.2.16}$$

A summary of the badness of raster scan with optimal weights is shown in Table 6.1.

l	Linear Model	Sublinear Model
1	$0.707\sqrt{w}n^{\frac{3}{4}}$	$(0.711 + O(1))\sqrt{w}n^{\frac{7}{8}}$
2	$0.500\sqrt{w}n^{\frac{3}{4}}$	$(0.456 + O(1))\sqrt{w}n^{\frac{7}{8}}$
3	$0.408\sqrt{w}n^{\frac{3}{4}}$	$(0.348 + O(1))\sqrt{w}n^{\frac{7}{8}}$
4	$0.355\sqrt{w}n^{\frac{3}{4}}$	$(0.286 + O(1))\sqrt{w}n^{\frac{7}{8}}$
5	$0.316\sqrt{w}n^{\frac{3}{4}}$	$(0.245 + O(1))\sqrt{w}n^{\frac{7}{8}}$
6	$0.289\sqrt{w}n^{\frac{3}{4}}$	$(0.215 + O(1))\sqrt{w}n^{\frac{7}{8}}$
7	$0.267\sqrt{w}n^{\frac{3}{4}}$	$(0.193 + O(1))\sqrt{w}n^{\frac{7}{8}}$

Table 6.1: Summary of the badness of raster scan pixel ordering with optimal weights. Only the dominant term is shown.

Chapter 7

Analysis of MLP with Optimal Weights

We proceed with the analysis in the same way as in Chapter 4, analyzing the intermediate and the last levels separately. Note that the analysis of the intermediate levels is exactly the same as in Chapter 4, since the implicit weights of the intermediate levels are already optimal as we will now show. The notation remains unchanged. n is the number of pixels in the image; c the total number of neighbors used for prediction; d_j is the distance of neighbor j.

7.1 The intermediate levels

$$\mathcal{B}'_{MLP-int} = \sum_{\text{all pixels}} \sqrt{\frac{1}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}}. \tag{7.1.1}$$

Since all neighbors of each prediction have the same distance d from the predicted pixel,

$$\mathcal{B}'_{MLP-int} = \sum_{\text{all pixels}} \sqrt{\frac{1}{c\frac{1}{f(d)}}}. \tag{7.1.2}$$

Partitioning the predictions according to levels and number of neighbors,

$$\mathcal{B}'_{MLP-int} = \sum_{k=v}^{m} \left[\sum_{c=1}^{4} n_{k,c} \sqrt{\frac{f(d_k)}{c}} \right] \tag{7.1.3}$$

which is the same expression worked upon in Section 4.1.2 and 4.1.3. We summarize our previous results Table 7.1 for convenience.

Model for $f(d)$	$\mathcal{B}'_{MLP-int}$
Linear	$\sqrt{u}\Big[\left(2^{-\frac{1}{2}}a_6 + \frac{\sqrt{2}}{8}\right) n$ $+ (a_7 + a_4)\, n^{\frac{1}{2}}$ $+ \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^v a_6\right)\left(2^{\frac{1-v}{4}}s^{-\frac{1}{4}}\right) n^{\frac{1}{4}}$ $+ \left(2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right)\Big]$
Sublinear	$\sqrt{u}\Big[\left(b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}}\right) n$ $+ \left(b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}}\right) n^{\frac{1}{2}}$ $+ \left(b_{10}2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}}$ $+ (b_2 - b_{10})\, 2^{\frac{1}{4}}\Big]$

Table 7.1: Badness of MLP intermediate levels.

7.2 The Last Level

$$\mathcal{B}'_{MLP-last} = \sum_{\text{all pixels}} \sqrt{\frac{1}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}}. \tag{7.2.4}$$

Since each prediction uses neighbors of distance 1 and 3 only,

$$\mathcal{B}'_{MLP-last} = \sum_{\text{all pixels}} \sqrt{\frac{1}{c_1 \frac{1}{f(1)} + c_3 \frac{1}{f(3)}}}. \tag{7.2.5}$$

Partitioning the predictions according to number of neighbors,

$$\mathcal{B}'_{MLP-last} = \sum_{c=1}^{16} n_{last,c} \sqrt{\frac{1}{c_1 \frac{1}{f(1)} + c_3 \frac{1}{f(3)}}} \tag{7.2.6}$$

$$= \sum_{c=1}^{16} n_{last,c} \sqrt{\frac{f(1)f(3)}{c_1 f(3) + c_3 f(1)}}. \tag{7.2.7}$$

Substituting the values for $n_{last,c}, c_1$ and $c3$ from Table 4.2,

$$\begin{aligned}
\mathcal{B}'_{MLP-last} =& 2\sqrt{\frac{f(1)f(3)}{2f(3)+4f(1)}} + 4\sqrt{\frac{f(1)f(3)}{3f(3)+5f(1)}} + 4\sqrt{\frac{f(1)f(3)}{3f(3)+6f(1)}} \\
&+ \left[2 + 2\left(\sqrt{n} - 6\right)\right]\sqrt{\frac{f(1)f(3)}{3f(3)+7f(1)}} + 4\sqrt{\frac{f(1)f(3)}{4f(3)+8f(1)}} \\
&+ 2\left(\sqrt{n} - 6\right)\sqrt{\frac{f(1)f(3)}{4f(3)+9f(1)}} + 2\sqrt{\frac{f(1)f(3)}{4f(3)+10f(1)}} \\
&+ 2\left(\sqrt{n} - 6\right)\sqrt{\frac{f(1)f(3)}{4f(3)+11f(1)}} \\
&+ \frac{1}{2}\left(\sqrt{n} - 6\right)^2 \sqrt{\frac{f(1)f(3)}{4f(3)+12f(1)}}.
\end{aligned} \tag{7.2.8}$$

At this point we can choose to use the linear model ($f(d) = ud$) or the sublinear model ($f(d) = u\sqrt{d}$) for our analysis. We show the analysis for both in the next two subsections.

7.2.1 Analysis Using Linear Model for $f(d)$

Substituting $f(d) = ud$ (2.3.18),

$$\begin{aligned}
\mathcal{B}'_{MLP-last,\ lin} =& 2\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{2(u\cdot 3)+4(u\cdot 1)}} + 4\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{3(u\cdot 3)+5(u\cdot 1)}} \\
&+ 4\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{3(u\cdot 3)+6(u\cdot 1)}} + \left(2\sqrt{n}-10\right)\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{3(u\cdot 3)+7(u\cdot 1)}} \\
&+ 4\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{4(u\cdot 3)+8(u\cdot 1)}} + \left(2\sqrt{n}-12\right)\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{4(u\cdot 3)+9(u\cdot 1)}} \\
&+ 2\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{4(u\cdot 3)+10(u\cdot 1)}} + \left(2\sqrt{n}-12\right)\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{4(u\cdot 3)+11(u\cdot 1)}} \\
&+ \left(\frac{1}{2}n-6\sqrt{n}+18\right)\sqrt{\frac{(u\cdot 1)(u\cdot 3)}{4(u\cdot 3)+12(u\cdot 1)}} \qquad (7.2.9)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}'_{MLP-last,\ lin} =& 2\sqrt{\frac{3u}{10}} + 4\sqrt{\frac{3u}{14}} + 4\sqrt{\frac{u}{5}} \\
&+ \left(2\sqrt{n}-10\right)\sqrt{\frac{3u}{16}} + 4\sqrt{\frac{3u}{20}} \\
&+ \left(2\sqrt{n}-12\right)\sqrt{\frac{u}{7}} + 2\sqrt{\frac{3u}{22}} \\
&+ \left(2\sqrt{n}-12\right)\sqrt{\frac{3u}{23}} \\
&+ \left(\frac{1}{2}n-6\sqrt{n}+18\right)\sqrt{\frac{u}{8}} \qquad (7.2.10)
\end{aligned}$$

$$\begin{aligned}\mathcal{B}'_{MLP-last,\ lin} = &\left(\frac{1}{4\sqrt{2}}\right)\sqrt{u}n + \left(\frac{\sqrt{3}}{2} + 2\sqrt{\frac{1}{7}} + 2\sqrt{\frac{3}{23}} - 6\sqrt{\frac{}{8}}\right)\sqrt{un} \\ &+ \left(2\sqrt{\frac{3}{10}} + 4\sqrt{\frac{3}{14}} + 4\sqrt{\frac{1}{5}} - 10\frac{\sqrt{3}}{4} + 4\sqrt{\frac{3}{20}}\right. \\ &\left. - 12\sqrt{\frac{1}{7}} + 2\sqrt{\frac{3}{22}} - 12\sqrt{\frac{3}{23}} + 18\sqrt{\frac{1}{8}}\right)\sqrt{u}. \end{aligned} \tag{7.2.11}$$

Replacing the constant expressions in square brackets with a_9, a_{10} and a_{11} respectively,

$$\mathcal{B}'_{MLP-last,\ lin} = \sqrt{u}\left[a_9 n + a_{10}\sqrt{n} + a_{11}\right] \tag{7.2.12}$$

where constants a_9, a_{10} and a_{11} are:

$$a_9 = \frac{1}{4\sqrt{2}} \qquad \approx 0.1768 \tag{7.2.13}$$

$$a_{10} = \frac{\sqrt{3}}{2} + 2\sqrt{\frac{1}{7}} + 2\sqrt{\frac{3}{23}} - 6\sqrt{\frac{}{8}} \tag{7.2.14}$$

$$\begin{aligned} a_{11} = &2\sqrt{\frac{3}{10}} + 4\sqrt{\frac{3}{14}} + 4\sqrt{\frac{1}{5}} - 10\frac{\sqrt{3}}{4} + 4\sqrt{\frac{3}{20}} \\ &- 12\sqrt{\frac{1}{7}} + 2\sqrt{\frac{3}{22}} - 12\sqrt{\frac{3}{23}} + 18\sqrt{\frac{1}{8}}. \end{aligned} \tag{7.2.15}$$

7.2.2 Analysis Using Sublinear Model for $f(d)$

Substituting $f(d) = u\sqrt{d}$ (2.3.28),

$$\begin{aligned}
\mathcal{B}'_{MLP-last,\ sub} = \ & 2\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{2(u\cdot\sqrt{3})+4(u\cdot\sqrt{1})}} + 4\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{3(u\cdot\sqrt{3})+5(u\cdot\sqrt{1})}} \\
& + 4\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{3(u\cdot\sqrt{3})+6(u\cdot\sqrt{1})}} \\
& + \left(2\sqrt{n}-10\right)\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{3(u\cdot\sqrt{3})+7(u\cdot\sqrt{1})}} \\
& + 4\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{4(u\cdot\sqrt{3})+8(u\cdot\sqrt{1})}} \\
& + \left(2\sqrt{n}-12\right)\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{4(u\cdot\sqrt{3})+9(u\cdot\sqrt{1})}} \\
& + 2\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{4(u\cdot\sqrt{3})+10(u\cdot\sqrt{1})}} \\
& + \left(2\sqrt{n}-12\right)\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{4(u\cdot\sqrt{3})+11(u\cdot\sqrt{1})}} \\
& + \left(\frac{1}{2}n-6\sqrt{n}+18\right)\sqrt{\frac{(u\cdot\sqrt{1})(u\cdot\sqrt{3})}{4(u\cdot\sqrt{3})+12(u\cdot\sqrt{1})}}
\end{aligned} \tag{7.2.16}$$

$$\begin{aligned}\mathcal{B}'_{MLP-last,\ sub} = \ & 2\sqrt{\frac{u\sqrt{3}}{2\sqrt{3}+4}} + 4\sqrt{\frac{u\sqrt{3}}{3\sqrt{3}+5}} \\ & + 4\sqrt{\frac{u\sqrt{3}}{3\sqrt{3}+6}} + \left(2\sqrt{n}-10\right)\sqrt{\frac{u\sqrt{3}}{3\sqrt{3}+7}} \\ & + 4\sqrt{\frac{u\sqrt{3}}{4\sqrt{3}+8}} + \left(2\sqrt{n}-12\right)\sqrt{\frac{u\sqrt{3}}{4\sqrt{3}+9}} \\ & + 2\sqrt{\frac{u\sqrt{3}}{4\sqrt{3}+10}} + \left(2\sqrt{n}-12\right)\sqrt{\frac{u\sqrt{3}}{4\sqrt{3}+11}} \\ & + \left(\frac{1}{2}n - 6\sqrt{n} + 18\right)\sqrt{\frac{u\sqrt{3}}{4\sqrt{3}+12}}. \end{aligned} \tag{7.2.17}$$

Collecting n-terms together,

$$\begin{aligned}\mathcal{B}'_{MLP-last,\ sub} = \ & \left(\frac{1}{2}\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+12}}\right)\sqrt{u}n \\ & + \left(2\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+7}} + 2\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+9}} + 2\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+11}}\right. \\ & \quad \left. - 6\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+12}}\right)\sqrt{un} \\ & + \left(2\sqrt{\frac{\sqrt{3}}{2\sqrt{3}+4}} + 4\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+5}} + 4\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+6}}\right. \\ & \quad - 10\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+7}} + 4\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+8}} - 12\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+9}} \\ & \quad \left. + 2\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+10}} - 12\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+11}} + 18\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+12}}\right)\sqrt{u}. \end{aligned} \tag{7.2.18}$$

Replacing constant expressions in round brackets with b_{11}, b_{12} and b_{13},

$$\mathcal{B}'_{MLP-last,\ sub} = \sqrt{u}\left[b_{11}n + b_{12}\sqrt{n} + b_{13}\right] \tag{7.2.19}$$

where constants b_{11}, b_{12} and b_{13} are:

$$b_{11} = \frac{1}{2}\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+12}} \qquad \approx 0.1513 \tag{7.2.20}$$

$$b_{12} = 2\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+7}} + 2\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+9}} + 2\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+11}} - 6\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+12}} \tag{7.2.21}$$

$$\begin{aligned} b_{13} = {} & 2\sqrt{\frac{\sqrt{3}}{2\sqrt{3}+4}} + 4\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+5}} + 4\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+6}} \\ & - 10\sqrt{\frac{\sqrt{3}}{3\sqrt{3}+7}} + 4\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+8}} - 12\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+9}} \\ & + 2\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+10}} - 12\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+11}} + 18\sqrt{\frac{\sqrt{3}}{4\sqrt{3}+12}}. \end{aligned} \tag{7.2.22}$$

7.3 The Badness of MLP with Optimal Weights

7.3.1 Analysis Using Linear Model for $f(d)$

Combining expressions for $\mathcal{B}'_{MLP-int}$ (from Table 7.1) and $\mathcal{B}'_{MLP-last}$ (Equation 7.2.12),

$$\begin{aligned}
\mathcal{B}'_{MLP,\ lin} = \ & sz' \\
& + \sqrt{u}\Bigg[\left(2^{-\frac{1}{2}}a_6 + \frac{\sqrt{2}}{8}\right)n \\
& \qquad + (a_7 + a_4)\, n^{\frac{1}{2}} \\
& \qquad + \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^v a_6\right)\left(2^{\frac{1-v}{4}}s^{-\frac{1}{4}}\right)n^{\frac{1}{4}} \\
& \qquad + \left(2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right)\Bigg] \\
& + \sqrt{u}\Big[a_9 n + a_{10}n^{\frac{1}{2}} + a_{11}\Big].
\end{aligned} \tag{7.3.23}$$

Grouping n-terms together,

$$\begin{aligned}
\mathcal{B}'_{MLP,\ lin} = \ & sz' \\
& + \sqrt{u}\Bigg[\left(2^{-\frac{1}{2}}a_6 + \frac{\sqrt{2}}{8} + a_9\right)n \\
& \qquad + (a_7 + a_4 + a_{10})\, n^{\frac{1}{2}} \\
& \qquad + \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^v a_6\right)\left(2^{\frac{1-v}{4}}s^{-\frac{1}{4}}\right)n^{\frac{1}{4}} \\
& \qquad + \left(2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8 + a_{11}\right)\Bigg].
\end{aligned} \tag{7.3.24}$$

Substituting $u = \frac{w}{\sqrt{n}}$ (2.3.20),

$$\begin{aligned}
\mathcal{B}'_{MLP,\ lin} = \ & sz' \\
& + \left[\left(2^{-\frac{1}{2}}a_6 + \frac{\sqrt{2}}{8} + a_9\right)\sqrt{w}n^{\frac{3}{4}}\right. \\
& + \left(a_7 + a_4 + a_{10}\right)\sqrt{w}n^{\frac{1}{4}} \\
& + \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^v a_6\right)\left(2^{\frac{1-v}{4}}s^{-\frac{1}{4}}\right)\sqrt{w} \\
& + \left.\left(2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8 + a_{11}\right)\sqrt{w}n^{-\frac{1}{4}}.\right. \qquad (7.3.25)
\end{aligned}$$

7.3.2 Analysis Using Sublinear Model for $f(d)$

Combining expressions for $\mathcal{B}'_{MLP-int}$ (from Table 7.1) and $\mathcal{B}'_{MLP-last}$ (Equation 7.2.19),

$$\begin{aligned}
\mathcal{B}'_{MLP,\ sub} = \ & sz' \\
& + \sqrt{u}\left[\left(b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}}\right)n\right. \\
& \quad + \left(b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}}\right)n^{\frac{1}{2}} \\
& \quad + \left(b_{10}2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}} \\
& \quad + \left.\left(b_2 - b_{10}\right)2^{\frac{1}{4}}\right] \\
& + \sqrt{u}\left[b_{11}n + b_{12}n^{\frac{1}{2}} + b_{13}\right]. \qquad (7.3.26)
\end{aligned}$$

Grouping n-terms together,

$$\begin{aligned}\mathcal{B}'_{MLP,\ sub} = \ & sz' \\ & + \sqrt{u}\Bigg[\left(b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}} + b_{11}\right) n \\ & \quad + \left(b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}} + b_{12}\right) n^{\frac{1}{2}} \\ & \quad + \left(b_{10} 2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)\left(\frac{n}{s}\right)^{\frac{1}{8}} \\ & \quad + \left(b_2 2^{\frac{1}{4}} - b_{10}2^{\frac{1}{4}} + b_{13}\right)\Bigg]. \end{aligned} \tag{7.3.27}$$

Substituting $u = \frac{w}{\sqrt[4]{n}}$ (2.3.30),

$$\begin{aligned}\mathcal{B}'_{MLP,\ sub} = \ & sz' \\ & + \left(b_8 2^{-\frac{3}{4}} + \frac{1}{8}2^{\frac{1}{4}} + b_{11}\right)\sqrt{w}n^{\frac{7}{8}} \\ & + \left(b_9 2^{-\frac{1}{4}} + b_1 2^{-\frac{1}{4}} + b_{12}\right)\sqrt{w}n^{\frac{3}{8}} \\ & + \left(b_{10} 2^{\frac{1-v}{8}} - b_8 s 2^{\frac{1+7v}{8}} - b_9\sqrt{s}2^{\frac{1+3v}{8}}\right)s^{\frac{1}{8}}\sqrt{w} \\ & + \left(b_2 2^{\frac{1}{4}} - b_{10}2^{\frac{1}{4}} + b_{13}\right)\sqrt{w}n^{-\frac{1}{8}}. \end{aligned} \tag{7.3.28}$$

The badness of MLP with optimal weights is summarized in Table 7.2. There is some improvement over the previously computed one without the optimal weights; however, it is still worse than raster scan at $l = 2$ with optimal weights.

	Linear Model	Sublinear Model
$\mathcal{B}_{MLP}$	$0.608\sqrt{wn}^{\frac{3}{4}}$	$0.473\sqrt{wn}^{\frac{7}{8}}$
$\mathcal{B}'_{MLP}$	$0.587\sqrt{wn}^{\frac{3}{4}}$	$0.469\sqrt{wn}^{\frac{7}{8}}$
$\mathcal{B}'_{rasterscan,l=2}$	$0.500\sqrt{wn}^{\frac{3}{4}}$	$0.456\sqrt{wn}^{\frac{7}{8}}$

Table 7.2: Summary of the badness of MLP pixel ordering with optimal weights. Only the dominant term is shown.

Chapter 8

Conclusion

In this thesis, we have attempted to quantify the pixel ordering problem by proposing a mathematical badness measure for evaluating pixel ordering with a particular pixel choice. This measure is based on the statistical variance of prediction errors which is used to estimate the number of bits required to encode a prediction error.

Using this measure, we have shown how two well-known pixel ordering, MLP and raster scan, can be analyzed. The analysis has shown that MLP is generally worse than a raster scan pixel ordering that uses all neighbors up to a distance of 2. Using pixel choice with more neighbors for raster scan will improve its performance even more. A few variations of the MLP pixel ordering scheme are proposed and analyzed in the quest for a fairer comparison with raster scan and insight into the pixel ordering problem. Although these variations did perform better than standard MLP under our measure, they still lag behind raster scan. That raster scan is better than MLP can

be further substantiated by the zero order entropy values obtained through experimentation as tabulated in Table 8.1. In the experiment, the mean is used as the prediction function both in the standard MLP pixel ordering and in the raster scan pixel ordering using all neighbors up to a distance of 2.

Image	0-order Entropy	
	MLP	Raster Scan ($l = 2$)
lenna 512	4.81044	4.78031
barbara 512	5.77607	5.76520
goldhill 512	5.31637	5.22996
fruits 512	3.05386	2.94682
mri 256	5.36377	5.26937

Table 8.1: Zero-order entropy of prediction errors from MLP and raster scan pixel ordering. The mean of the neighbors' pixel values are used as the predictor. The entropy is calculated the error image which includes the unpredictable starting pixels.

The problem of how the weights of a prediction function might affect pixel ordering and pixel choice has also been discussed and our badness measure has been modified to evaluate pixel ordering under the condition of optimal weights. Using this improved measure, the raster scan and MLP pixel ordering are analyzed again. Their relative performance remains unchanged. The important results are summarized again in Table 8.2.

Future Research

Our two models for $f(d)$ are still very coarse. It should be noted that the logarithm curve fits the smaller distances of many images very well too. In this thesis, it is not used because of the difficulties it poses in the analysis of

	Linear Model	Sublinear Model
$\mathcal{B}'_{rasterscan,l=3}$	$0.408\sqrt{w}n^{\frac{3}{4}}$	$0.348\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{rasterscan,l=3}$	$0.441\sqrt{w}n^{\frac{3}{4}}$	$0.355\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{GMLP20}$	$0.458\sqrt{w}n^{\frac{3}{4}}$	-
$\mathcal{B}'_{rasterscan,l=2}$	$0.500\sqrt{w}n^{\frac{3}{4}}$	$0.456\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{rasterscan,l=2}$	$0.527\sqrt{w}n^{\frac{3}{4}}$	$0.464\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}'_{MLP}$	$0.587\sqrt{w}n^{\frac{3}{4}}$	$0.469\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{GMLP6}$	$0.588\sqrt{w}n^{\frac{3}{4}}$	-
$\mathcal{B}_{MLP}$	$0.608\sqrt{w}n^{\frac{3}{4}}$	$0.473\sqrt{w}n^{\frac{7}{8}}$

Table 8.2: Summary of the important results proved in this thesis. Only the dominant term is shown. $\mathcal{B}$ denotes original badness measure and $\mathcal{B}'$ the badness measure with optimal weights.

MLP; however, it might be tractable for other pixel orderings.

Another class of pixel ordering that we would like to analyze with our badness measure are those defined by *space filling curves*. These curves are like paths that traverse each pixel in the image exactly once and hence can be used as pixel orderings. Different curves have different characteristics like whether the paths explore the image in small regions. Such characteristics are hope to exploit local similarities.

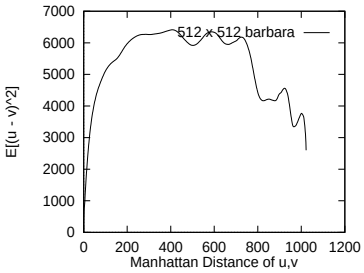
Lastly, our badness measure does not differentiate between the direction of neighbors relative to the predicted pixel. For example, a distance 1 neighbor that is directly above the predicted pixel is no different from a distance 1 neighbor that is directly on the left of the predicted pixel. Whether the direction of neighbors have any significant effect on compression is another topic for future research.

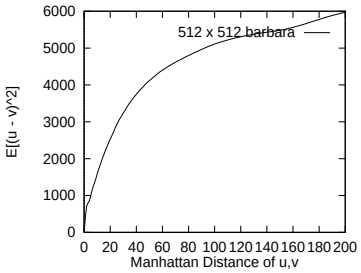
Appendix A

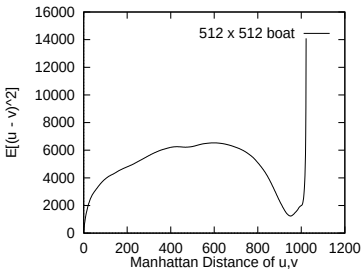
More $E[(p_u - p_v)^2]$ Plots

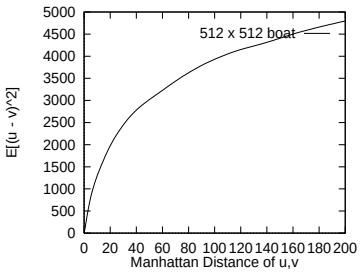
Each of the following rows show the image used, its $E[(p_u - p_v)^2]$ plot over the full range of distances and the same plot over smaller distances.

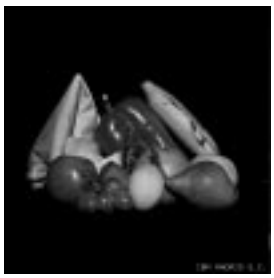
A.1 Plots for 512×512 Images

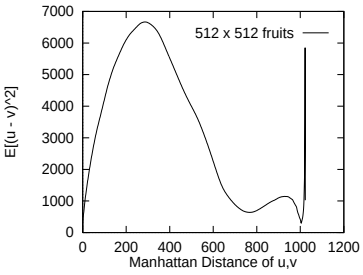


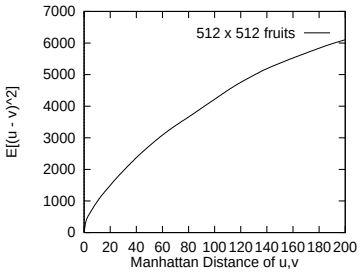


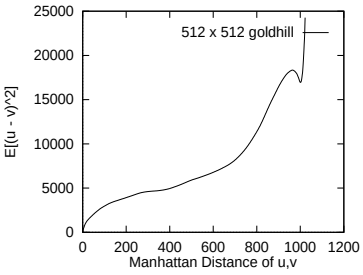


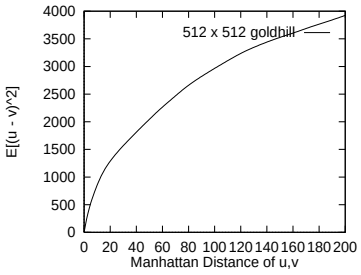


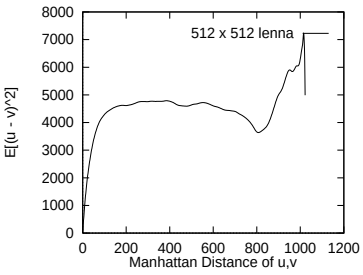


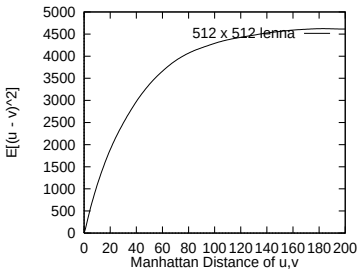

512 x 512 fruits
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000 7000
0 200 400 600 800 1000 1200

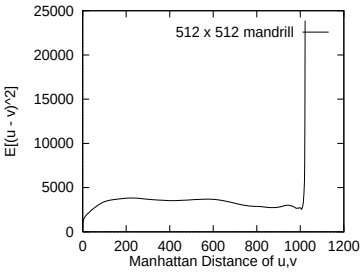

512 x 512 fruits
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000 7000
0 20 40 60 80 100 120 140 160 180 200

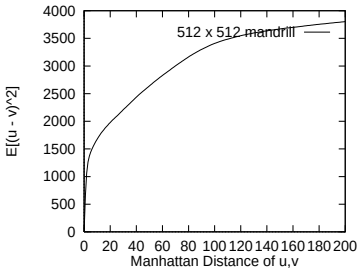

512 x 512 goldhill
E[(u - v)^2]
Manhattan Distance of u,v
0 5000 10000 15000 20000 25000
0 200 400 600 800 1000 1200

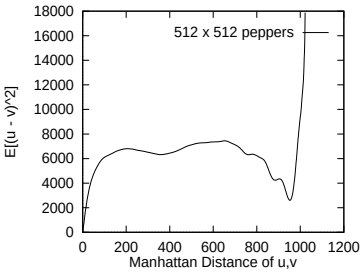

512 x 512 goldhill
E[(u - v)^2]
Manhattan Distance of u,v
0 500 1000 1500 2000 2500 3000 3500 4000
0 20 40 60 80 100 120 140 160 180 200

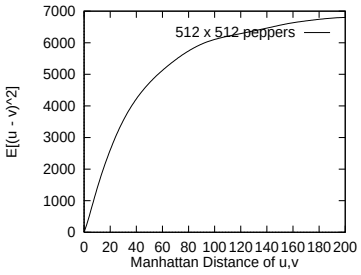

512 x 512 lenna
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000 7000 8000
0 200 400 600 800 1000 1200

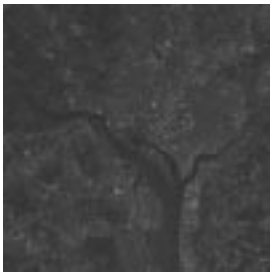

512 x 512 lenna
E[(u - v)^2]
Manhattan Distance of u,v
0 500 1000 1500 2000 2500 3000 3500 4000 4500 5000
0 20 40 60 80 100 120 140 160 180 200

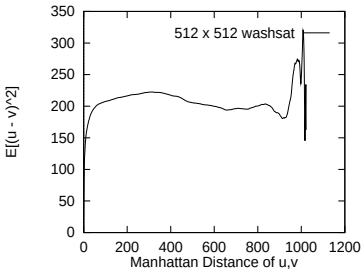

512 x 512 mandrill
E[(u - v)^2]
Manhattan Distance of u,v
0 5000 10000 15000 20000 25000
0 200 400 600 800 1000 1200

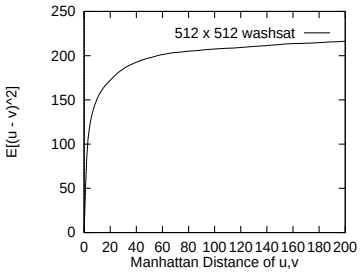

512 x 512 mandrill
E[(u - v)^2]
Manhattan Distance of u,v
0 500 1000 1500 2000 2500 3000 3500 4000
0 20 40 60 80 100 120 140 160 180 200

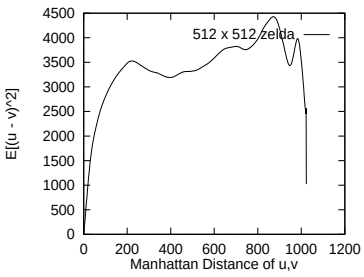

512 x 512 peppers
E[(u - v)^2]
Manhattan Distance of u,v
0 2000 4000 6000 8000 10000 12000 14000 16000 18000
0 200 400 600 800 1000 1200

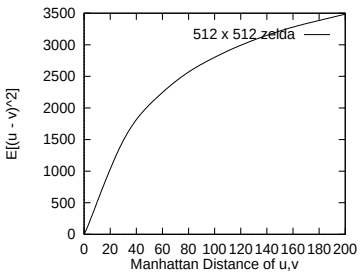

512 x 512 peppers
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000 7000
0 20 40 60 80 100 120 140 160 180 200



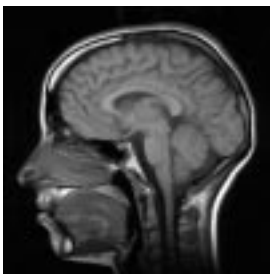


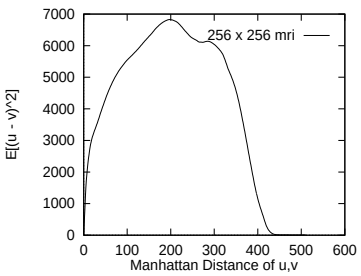


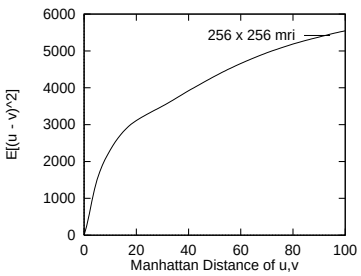


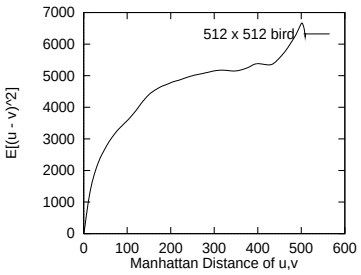


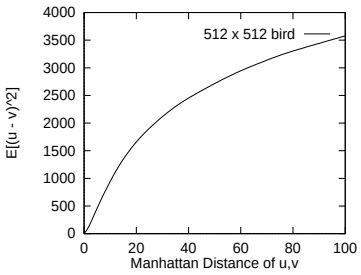
A.2 Plots for 256×256 Images

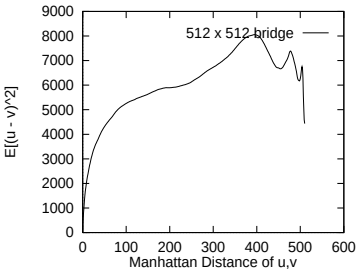


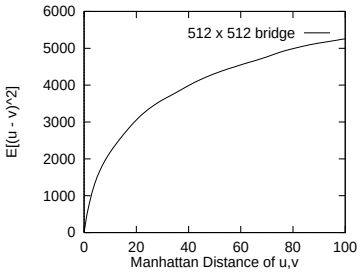


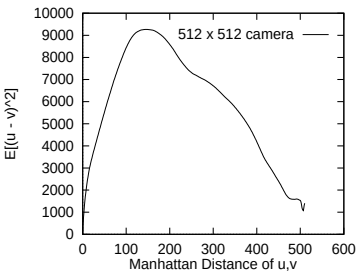


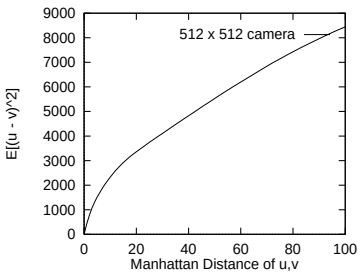


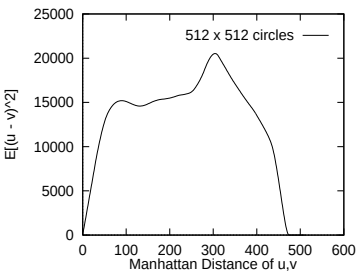


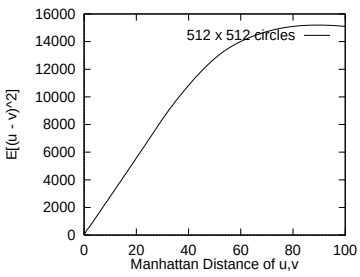

512 x 512 bridge
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000 7000 8000 9000
0 100 200 300 400 500 600

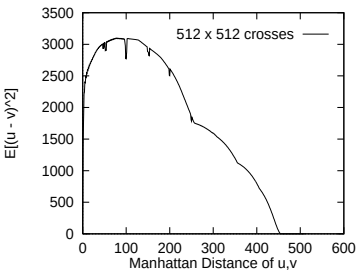

512 x 512 bridge
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000
0 20 40 60 80 100

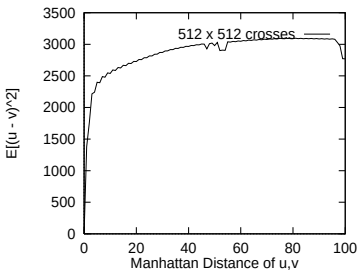

512 x 512 camera
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000 7000 8000 9000 10000
0 100 200 300 400 500 600

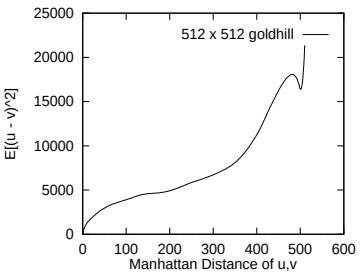

512 x 512 camera
E[(u - v)^2]
Manhattan Distance of u,v
0 1000 2000 3000 4000 5000 6000 7000 8000 9000
0 20 40 60 80 100


512 x 512 circles
E[(u - v)^2]
Manhattan Distance of u,v
0 5000 10000 15000 20000 25000
0 100 200 300 400 500 600


512 x 512 circles
E[(u - v)^2]
Manhattan Distance of u,v
0 2000 4000 6000 8000 10000 12000 14000 16000
0 20 40 60 80 100


512 x 512 crosses
E[(u - v)^2]
Manhattan Distance of u,v
0 500 1000 1500 2000 2500 3000 3500
0 100 200 300 400 500 600


512 x 512 crosses
E[(u - v)^2]
Manhattan Distance of u,v
0 500 1000 1500 2000 2500 3000 3500
0 20 40 60 80 100


512 x 512 goldhill
E[(u - v)^2]
Manhattan Distance of u,v
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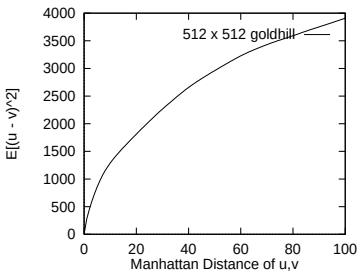
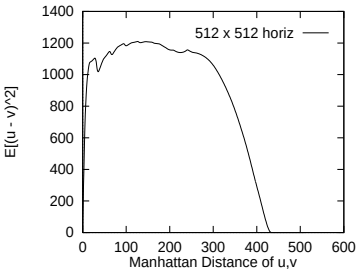
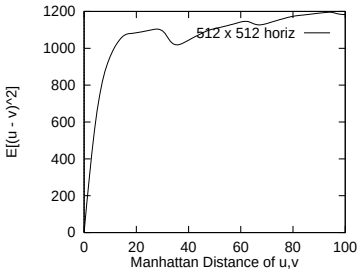
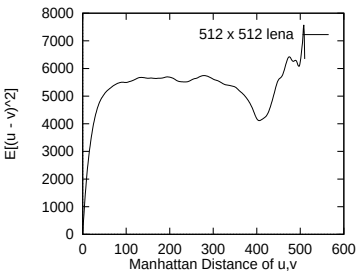
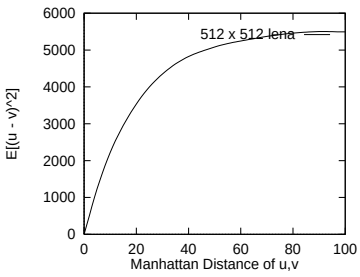
0 100 200 300 400 500 600


512 x 512 goldhill
E[(u - v)^2]
Manhattan Distance of u,v
0 500 1000 1500 2000 2500 3000 3500 4000
0 20 40 60 80 100

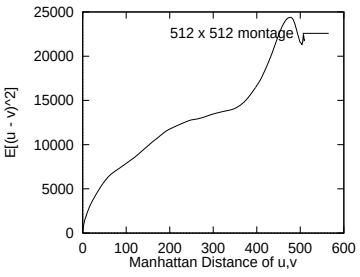

512 x 512 horiz
E[(u - v)^2]
Manhattan Distance of u,v

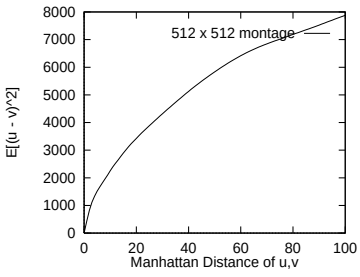

512 x 512 horiz
E[(u - v)^2]
Manhattan Distance of u,v

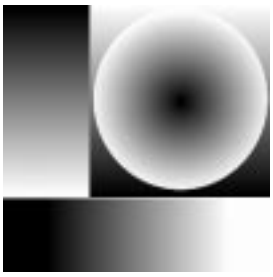

512 x 512 lena
E[(u - v)^2]
Manhattan Distance of u,v

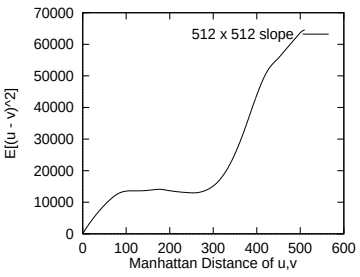

512 x 512 lena
E[(u - v)^2]
Manhattan Distance of u,v

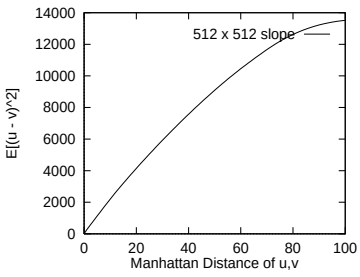
Hello World

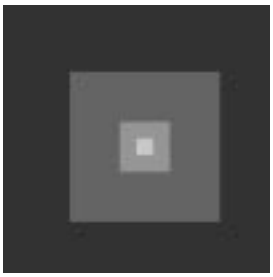

512 x 512 montage
E[(u - v)^2]
Manhattan Distance of u,v

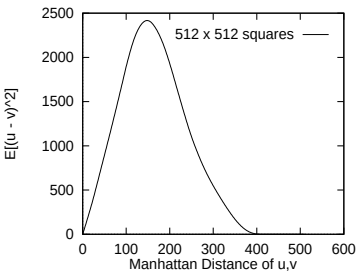

512 x 512 montage
E[(u - v)^2]
Manhattan Distance of u,v

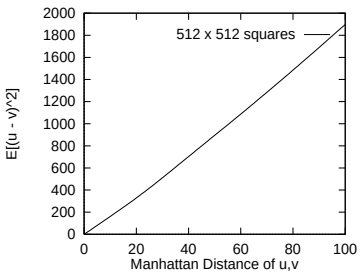



512 x 512 slope
E[(u - v)^2]
Manhattan Distance of u,v

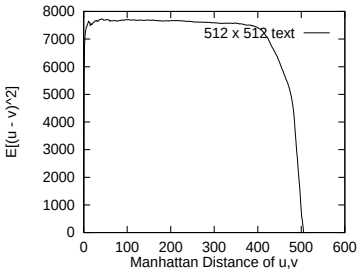

512 x 512 slope
E[(u - v)^2]
Manhattan Distance of u,v

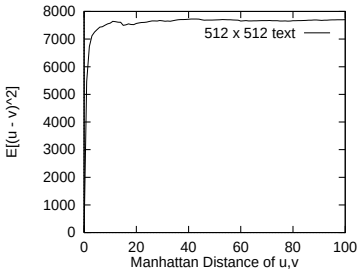



512 x 512 squares
E[(u - v)^2]
Manhattan Distance of u,v


512 x 512 squares
E[(u - v)^2]
Manhattan Distance of u,v

MT=LEVEL
Espinet3/home/u/rjkroeger/vf
Espinet3/home/u/rjkroeger/vf
Makefile compress.c f
Makefile~ display.c fr
Espinet3/home/u/rjkroeger/vf
Espinet3/home/u/rjkroeger/vf
Makefile compress.c f
Makefile~ display.c fr
Espinet3/home/u/rjkroeger/vf
Makefile display.c tm
Makefile~ fileio.c tm
compress.c fractal.h tm
Espinet3/home/u/rjkroeger/vf
Espinet3/home/u/rjkroeger/vf
Espinet3/home/u/rjkroeger/vf


512 x 512 text
E[(u - v)^2]
Manhattan Distance of u,v
0
1000
2000
3000
4000
5000
6000
7000
8000
0
100
200
300
400
500
600


512 x 512 text
E[(u - v)^2]
Manhattan Distance of u,v
0
1000
2000
3000
4000
5000
6000
7000
8000
0
20
40
60
80
100

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