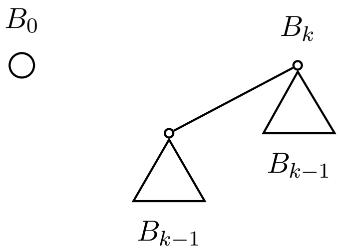
Lecture 11 Binomial Heaps

Feb 12, 2014

A **Binomial tree** B_k of degree k is defined recursively as follows.



Note that the order of the child nodes are significant.

Properties of a Binomial Tree

1. B_k has 2^k nodes.
2. B_k has height k.
3. B_k has $\binom{k}{i}$ at depth $i = 0, \ldots, k$.
4. Root of B_k has degree k which is maximum over all nodes.
5. The subtree rooted at the i-th child of the root of B_k is a binomial tree B_i.

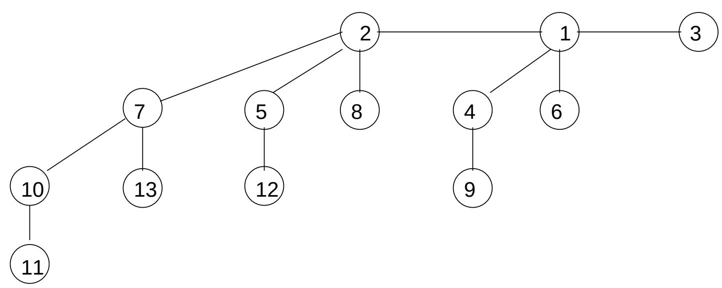
$\Rightarrow$ Maximum degree in an n-node binomial tree is $\lg n$.

A **Binomial Heap** is a set of binomial trees such that

1. each binomial tree is heap-ordered,
2. there is at most one B_k for a given k.

$\Rightarrow$ The binomial heap for n items contains one binomial tree for each 1-bit in the binary representation of n. If the i-th bit is set, then the corresponding binomial tree is B_i, where the least significant bit occurs at $i = 0$.

Example. The binomial heap for 13 items (binary 1101) is shown below.



The root nodes of all the binomial trees in the binomial heap are chained together in a doubly-linked circular list called the *root list*. The children of each node are also chained together in a doubly-linked circular list to facilitate merging.

Minimum(H) Iterate through the root list to find the minumum root. Root list has at most $\lg n$ nodes.

Union(H_1, H_2) Merge the two root list in order of root degrees. Iterate through merged root list and merge the binomial trees analogous to binary addition. Merging two binomal trees take $O(1)$ time. There are $O(\lg n_1 + \lg n_2) = O(\lg(n_1 + n_2))$ trees to merge.

Insert(H, x) Make x a single node binomial heap and union with H.

ExtractMin(H) Find Minimum(H), remove minimum root, make its children into a new binomial heap, and union with H.

DecreaseKey(H, x, k) Update $x.key$ to k, and bubble x up to maintain heap order.

Delete(H, x) Decrease key of x to $-\infty$, and extract minimum.

Operation	Binary Heaps	Binomial Heaps	Fibonacci Heaps
Insert	$O(\lg n)$	$O(\lg n), O(1)^A$	$O(1)$
Minimum	$O(1)$	$O(\lg n)$	$O(1)$
ExtractMin	$O(\lg n)$	$O(\lg n)$	$O(\lg n)^A$
Delete	$O(\lg n)$	$O(\lg n)$	$O(\lg n)^A$
DecreaseKey	$O(\lg n)$	$O(\lg n)$	$O(1)^A$
Union	$O(n)$	$O(\lg n), O(1)^A$	$O(1)$